Instabilities of Radiative Convective Equilibrium with an Interactive Surface

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Moist radiative-convective equilibrium (RCE) is the equilibrium state of the atmosphere in the absence of lateral transport when radiation, convection, and water phase changes are taken into account. If cloud radiative feedbacks and lateral advection are neglected, we show that RCE is always linearly unstable to atmospheric moisture perturbations under the weak temperature gradient approximation, leading to a dry zone with large-scale descent and a moist zone with large-scale ascent. If the surface is allowed to be interactive, an air-sea radiative-convective instability develops, strongest for low surface temperature and heat capacity. This instability can couple with the purely atmospheric radiative-convective instability found by (Emanuel et al. 2014), leading to larger linear growth rates for the full instability. Finally, we show that this instability exhibits two types of dry biases: the dry instability grows faster than its moist counterpart and the dry zone occupies a larger area than the moist zone in the final state.

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1. Introduction

One-dimensional radiative-convective models are arguably the simplest representations of the tropical atmosphere. Among them, moist radiative-convective equilibrium (RCE) is the equilibrium state of the atmosphere in the absence of lateral transport when radiation, convection, and water phase changes are taken into account. It is not as well understood as its dry counterpart; yet it is an excellent approximation to planetary climates where the main greenhouse gas exist in their liquid and vapor phases (Ramanathan and Coakley 1978).

It is accepted that RCE is unique when the greenhouse gas profile and the microphysics of the atmosphere are held fixed, except when the insolation is too high, in which case the runaway greenhouse effect can occur (Ingersoll 1969; Pujol and North 2002). However, when a large-scale circulation is allowed to develop, RCE has been shown to be conditionally unstable and to exhibit multiple equilibria (Renno 1997; Raymond and Zeng 2000), even when homogeneous boundary conditions are imposed. Using a two-dimensional cloud resolving model, (Held et al. 1993) was the first one to show that convection could spontaneously self-organize into a small moist cluster with mean ascent, surrounded by a dry subsiding area. Since then, the presence and mechanisms of convective self-aggregation have been studied in 3D cloud-resolving models (Tompkins 2001; Jevanjee and Romps 2013; Wing and Emanuel 2014; Muller and Bony 2015) and general circulation models (Bretherton and Khairoutdinov 2015; Coppin and Bony 2015); on a f-plane where it systematically leads to cyclogenesis (Bretherton et al. 2005; Nolan et al. 2007; Khairoutdinov and Emanuel 2013) as well as on a β-plane, where it possibly generates Madden-Julian oscillation like disturbances (Arnold and Randall 2015). It is thought that the mechanisms that maintain self-aggregation are not the same one that make RCE unstable in the first place: (Wing and Emanuel 2014) found that the main trigger for the radiative-convective instability is the interaction between the clear-sky longwave radiation, the large-scale circulation and the water vapor distribution. Indeed, as the horizontal temperature gradients are very weak in the Tropics, moist static energy gradients are mainly generated by the variability of the atmospheric water vapor concentration. The weak temperature gradient (WTG) framework (Sobel et al. 2001) has proven to be very useful to isolate this instability and study the possibility of multiple radiative-convective equilibria (Sessions et al. 2010, 2015), especially in a single column (Sobel et al. 2007; Emanuel et al. 2014), where the effects of large-scale circulation are included by allowing large-scale vertical velocities to be generated.

The fact that RCE can go unstable has important climatic consequences. (Held et al. 1993; Bony et al. 2015; Bretherton et al. 2005; Nolan et al. 2007) showed that aggregated convection leads to a warmer and drier troposphere in models, while (Tobin et al. 2012) observationally established that self-aggregation tended to dry the free troposphere and increase the outgoing longwave radiation. The dry bias of the RCE instability was further shown by the simulations of (Wing and Emanuel 2014; Boos et al. 2015), where the dry instability develops first, and the dry zone occupy a larger area than the moist one in the final state. Finally, the long relaxation times of coupled atmosphere-ocean RCE make simulations with cloud resolving models too computationally expensive. For that reason, all the previous studies have been
realized with a fixed surface temperature, despite the importance of having an interactive surface to evaluate the RCE timescales (Cronin and Emanuel 2013) and develop instabilities to large-scale circulations for a wider range of surface temperature (Nilsson and Emanuel 1999).

This paper builds on the work of (Emanuel et al. 2014) and addresses two questions:

1. What are the different instabilities of RCE that can develop with an interactive surface?
2. What are the physical mechanisms of the dry bias of RCE instabilities?

It relies on key simplifying assumptions: We neglect the shortwave and the cloud feedbacks because we are mostly interested in the onset of the instability. We make the WTG approximation to focus on the key variable of the instability: the water vapor concentration. We also neglect convective gustiness, preventing the wind induced surface heat exchange effect, which can be important for self-aggregation. We first establish a minimal theoretical framework for the air-sea radiative-convective instability that occurs when the surface is allowed to be interactive, in section 2. We then study the full radiative-convective instability by allowing the atmosphere to have a vertical structure in section 3. Finally, we use the MIT single column model (MIT SCM) to better understand the effect of the interactive surface and the dry bias of the instability in section 4.

2. Air-sea radiative-convective instability

2.1. One layer model of the tropical atmosphere

We follow the notations introduced in figure 1. The model is meant to be as simple as possible and comprises four parts; from bottom to top: the interactive surface (script s), the boundary layer (script b), the free troposphere (script a) and the tropopause (script t). The energy balance of the system is separated in two parts: convective (brown arrows) and radiative (red arrows). Only the theory relevant to the air-sea instability of RCE is presented here; readers interested in the full equations of the one layer model are referred to appendix A.

2.1.1. Convective parametrization

The convective parametrization is based on the dry static energy \( s \) (DSE)/moist static energy \( h \) (MSE) at each level, that we approximately define as:

\[
s \equiv c_p T + g z. \tag{1}\]

DSE/MSE is approximately conserved by an individual air parcel during dry/moist adiabatic processes (Arakawa and Schubert 1974). \( L_v \) is the latent heat of vaporization of water vapor, \( r \) is the mixing ratio of water vapor (defined as the ratio of the density of water vapor to the density of dry air), \( c_p \) is the specific heat capacity of dry air at constant pressure, \( T \) is the absolute temperature, \( g \) is the gravity constant and \( z \) the geopotential height. The convective fluxes are driven by the contrasts of MSE between the different levels of the model; they include:

- \( F_s \), the upwards MSE flux from the surface to the boundary layer, driven by the turbulence at the surface.
- \( F_a \), the cumulus MSE flux from the boundary layer to the free troposphere, driven by the cloudy downdrafts from the free troposphere to the boundary layer.
- \( F_b \), the subsidence MSE flux from the free-troposphere to the boundary layer.

As we wish to focus on the effects of water vapor perturbations, we make the weak temperature gradient (WTG) approximation (Sobel et al. 2001) which holds reasonably in the Tropics, where the Coriolis parameter is small and the convective heating is mainly controlled by the moisture content of the atmosphere. In our model, it means that the atmospheric temperatures can be approximated as constant, and that the diabatic heating will be compensated by the generation of large-scale vertical velocities.

2.1.2. Radiative parametrization

For the sake of simplicity, we assume that the insolation \( \sigma T_r^4 \) is constant and that the atmosphere is transparent to shortwave radiation. We have introduced the effective emission temperature \( T_e \) and the Stefan-Boltzmann constant \( \sigma \). The downsours solar flux is entirely absorbed by the surface, which is assumed to be a homogeneous blackbody of temperature \( T_s \) and prescribed heat capacity per surface unit \( C_s \). In the longwave domain, the atmosphere is assumed to be an isotropic gray body of temperature \( T_r \) and slab emissivity \( \varepsilon \), so that its emittance (emitted power per surface unit) is \( \varepsilon \sigma T_r^4 \) in every direction (upwards and downwards). Since we are mostly interested in the effects of water vapor, the main greenhouse gas in the Earth atmosphere, we approximate the emissivity \( \varepsilon \) to be a given increasing function of the free-tropospheric water vapor mixing ratio \( r_u \) (cf appendix A.1 for more details). The radiative heating of the surface and the free troposphere are thus respectively given in this model by:

\[ \dot{Q}_s = \sigma 
\]

\[ \dot{Q}_a = \varepsilon (r_u) [T_s^4 - 2T_r^4]. \tag{4} \]

2.1.3. Equations of the model

The equations of the model are the energy budgets for each component of the system. The interactive surface has a dynamic temperature \( T_s \), which evolves according to the heat equation:

\[
C_s \frac{\partial T_s}{\partial t} = \dot{Q}_s - F_s. \tag{5}
\]

Following (Raymond 1995) and (Emanuel 1995), the boundary layer is assumed to be in quasi-equilibrium and its net heating is zero in the cumulus regions. Thus, the convective fluxes must balance:

\[
F_s = F_a + F_b. \tag{6}
\]

In these regions, we follow (Neelin and Held 1987) and use the MSE budget in the free troposphere, in combination with the
WTG approximation, which guarantees that the temperature is steady, to write:

$$\frac{\partial h_a}{\partial t} \text{ WTG} = L_v \frac{\partial r_a}{\partial t} \approx \frac{g}{\Delta p} (F_a + F_b + Q_a).$$

To write (7) (in units W kg\(^{-1}\)), we have divided the energy fluxes (in units W m\(^{-2}\)) by the approximate mass per surface unit of the free troposphere \(\Delta p\) (in units kg m\(^{-2}\)), where \(\Delta p\) is the typical pressure thickness of the free troposphere. Combining equations (6) and (7) gives a simple equation for the evolution of the free-tropospheric mixing ratio:

$$\frac{L_v \Delta p}{g} \frac{\partial r_a}{\partial t} = F_a + Q_a.$$  \hspace{1cm} (8)

### 2.1.4. Radiative-convective equilibrium

We define RCE (denoted by overlines \(\overline{X}\)) as the steady state of equations (5) and (8) (\(\frac{\partial X}{\partial t} = 0\)) with no large-scale vertical velocity. In this model, RCE is unique for a given insolation \(\overline{F}_s\). As we can see on figure 2, the surface temperature in RCE \(\overline{T}_s\) increases monotonically with the insolation, which means that RCE is also unique for a given \(\overline{T}_s\). From now on, we take \(\overline{T}_s\) as the free parameter of the model to facilitate the physical interpretation of our results. From equations (6), (8) and (5), we obtain:

$$\overline{T}_s = \overline{Q}_s = -\overline{Q}_a.$$  \hspace{1cm} (9)

In RCE, the surface convective flux balances the radiative heating of the surface, equal to the radiative cooling of the atmosphere. A full computation of RCE in the one layer model is presented in appendix A.1.

### 2.2. Instability to water vapor and surface temperature perturbations

We are interested in the stability of RCE when the water vapor mixing ratio \(r_a\) and/or the sea surface temperature \(T_s\) are perturbed from their equilibrium values. The goal is to understand how the characteristics of this instability change with the two free parameters of this model: the RCE surface temperature \(\overline{T}_s\) and the heat capacity of the slab surface \(C_s\).

#### 2.2.1. Linear analysis

The linear stability analysis can be directly done by differentiating equations (5) and (8) with respect to the mixing ratio \(r_a\) and the surface temperature \(T_s\); it yields:

$$\frac{\partial}{\partial t} \left( \begin{array}{c} r_a' \\ T_s' \end{array} \right) = J \left( \begin{array}{c} r_a' \\ T_s' \end{array} \right).$$

We have denoted perturbation from RCE with primes \((X' = X - \overline{X})\), and \(J\) is the Jacobian matrix with the following coefficients:

\begin{align*}
J_{11} &= \frac{g}{L_v \Delta p} \frac{\partial Q_a}{\partial r_a} (< 0), \\
J_{12} &= \frac{g}{L_v \Delta p} \frac{\partial F_s}{\partial T_s} + \frac{\partial Q_a}{\partial T_s} (> 0), \\
J_{21} &= \frac{1}{C_s} \frac{\partial Q_a}{\partial r_a} (> 0), \\
J_{22} &= \frac{1}{C_s} \frac{\partial Q_a}{\partial T_s} + \frac{\partial F_s}{\partial T_s} (< 0).
\end{align*}

These coefficients are computed in more details in appendix A.2. The fact that the diagonal of the Jacobian: \(J_{11}, J_{22}\) is negative proves that the uncoupled system is always linearly stable. The coupled linear stability of the system can be found by looking at the eigenvalues \((\lambda_1, \lambda_2)\) of the Jacobian \(J\). From the trace and the determinant of \(J\), we respectively obtain the sum and the product of the eigenvalues:

$$\lambda_1 + \lambda_2 = J_{11} + J_{22} (< 0),$$

$$\lambda_1 \lambda_2 = J_{11} J_{22} - J_{12} J_{21} (< 0).$$

From equation (16), there will always be a positive eigenvalue (eg \(\lambda_1\)) and the coupled system is always linearly unstable. The linear instability grows like \(\exp(\lambda_1 t)\), and from equations (15) and (16), the growth rate is given by:

$$\lambda_1 = \frac{J_{11} + J_{22} + \sqrt{(J_{11} - J_{22})^2 + 4J_{12} J_{21}}}{2}.$$  \hspace{1cm} (17)

The reader is referred to appendix A.2 for a full expression of the growth rate \(\lambda_1\) as a function of the physical parameters of the problem. The growth rate is governed by the dimensionless surface heat capacity:

$$\tilde{C} \overset{\text{def}}{=} \frac{C_s T_{ref} g}{L_v r_{ref} \Delta p},$$  \hspace{1cm} (18)

defined as the ratio of the typical internal heat content of the slab surface, to the typical latent heat content of the atmosphere, \(r_{ref}\), the reference water vapor mixing ratio, is set to 1%, whereas \(T_{ref}\), the reference surface temperature, is set to 300K. In the limit of a fixed surface temperature, corresponding to a very large surface heat capacity \(\tilde{C} \gg 1\), the growth rate \(\lambda_1\) is zero. In the limit of a vanishingly small heat capacity \(\tilde{C} \ll 1\), it asymptotically reaches a constant \(\lambda_{1\infty} > 0\) given by equation (65). This asymptotic growth rate decreases with RCE surface temperature, because of the concavity of the slab emissivity \(\epsilon(r_a)\), and reaches zero when the slab emissivity \(\epsilon\) saturates to 1. Figure 3 shows that the growth rate strictly decreases with the two free parameters of the one layer model: the surface heat capacity and the RCE surface temperature.
2.2.2. Dry bias

We compare the result of our linear analysis to a numerical integration of the fully non-linear equations of the one-layer model in Figure 4. For short times, the linear approximations describe the solution very well. For very long times, the mixing ratio saturates for the moist branch of the instability (+10%) and completely dries out for the dry branch of the instability (−10%). For intermediate times, we notice the small dry bias of the instability compared to the first order linear approximation, by definition symmetric. Part of this dry bias is captured by the second order linear approximation, which means that we can (at least partially) explain it theoretically. We study the coefficients of the second order expansion in appendix A.2.3. A key coefficient to describe the dry bias of the instability is $H_{121}$ (70), the second order effect of the moisture perturbation on the surface heat equation (see (73)). This effect is negative (corresponding to a cooling/drying of the perturbation) because of the concavity of the emissivity as a function of the mixing ratio (see Figure 5), a feature exhibited by all the theoretical and observational functions $\varepsilon(r_a)$ (Brunt 1936; Brutsaert 1975):

$$\frac{\partial^2 \varepsilon}{\partial r_a^2} < 0. \quad (19)$$

The success of the linear analysis to describe the behavior of the one layer model helps us understand the physical mechanisms of the air-sea instability. Following figure 6, we start from RCE where by definition, the surface convective flux $F_s$, the radiative heating of the surface $Q_s$ and the radiative cooling of the atmosphere $Q_a$ are such that the surface and the atmosphere are in thermal equilibrium. We then introduce a positive water vapor perturbation in the atmosphere $r'_a > 0$. It increases the radiative cooling of the atmosphere $Q_a$, but also the radiative heating of the surface $Q_s$. The latter will slightly warm the surface up, as the surface convective flux $F_s$ is yet unchanged, which will introduce a positive surface temperature perturbation $T'_s > 0$. This perturbation will decrease the radiative heating $Q_s$ of the surface. However, it will also increase the surface convective flux $F_s$ and decrease the radiative cooling of the atmosphere $Q_a$, which both increase the MSE of the atmosphere. Equation (16) proves that the moistening effects of the atmosphere then overcome the cooling effects of the surface, and the perturbation $r'_a$ will be reinforced. This leads to a positive feedback between water vapor and surface temperature perturbations, and explains the occurrence of the air-sea instability, when the surface is allowed to be interactive. If the column becomes unstable, its MSE will increase, leading to a large-scale ascent according to the W TG approximation. According to mass conservation, it will generate a large-scale descent of the surrounding air, and introduce a negative MSE perturbation at this location $(r'_a, T'_s)$, leading to a drying and cooling instability. Eventually, we will have high MSE regions, where the air is moist and the surface warm, adjacent to low MSE regions, where the air is dry and the surface cool. Note that as the initial argument is based on linear theory, it is symmetrical and the signs can all be reversed to prove that an initially negative water vapor perturbation in the atmosphere $r'_a < 0$ leads to a drying and cooling air-sea instability. Finally, this physical mechanism is one-dimensional and neglects the horizontal advection of moisture in the atmosphere and the horizontal and vertical advection of heat in the ocean. It will thus only apply for reasonably short timescales (typically less than a month).
3. Full radiative-convective instability

3.1. Two layer model of the tropical atmosphere

Following the notations of figure 7, we generalize the one layer model defined in section 2.1 by dividing the free-troposphere in two layers of equal mass per surface unit (\(\Delta Z_i\)) : the lower troposphere (subscript 1) and the upper troposphere (subscript 2). We add the subscripts m for the mid-troposphere. The seven convective fluxes of this model are:

- \(F_s\), the surface convective flux.
- \(F_m\), the MSE flux produced by shallow convection, from the boundary layer to the lower troposphere. It ensures that the latter is radiatively cooling. We parametrize this flux as a fixed fraction \((1 - \alpha)\) of the surface flux.
- \(F_1\), the cumulus flux from the boundary layer to the lower troposphere.
- \(F_{b1}\), the subsidence MSE flux from the lower troposphere to the boundary layer.
- \(F_2\), the cumulus flux from the mid-troposphere to the upper troposphere.
- \(F_{b2}\), the subsidence MSE flux from the upper troposphere to the lower troposphere.
- \(F_m\), the gross moist stability which allows the convecting air column to exchange MSE with the large-scale circulation (Yu et al. 1998; Neelin and Held 1987; Yu and Neelin 1997).

The radiative heatings of the surface, the lower troposphere and the upper troposphere, are respectively given by:

\[
\dot{Q}_s = \sigma[\varepsilon(T_s^4 - T_{s1}^4)],
\]

\[
\dot{Q}_1 = \varepsilon_1[\sigma(T_s^4 - 2T_{s1}^4 + \varepsilon_2T_{s2}^4)],
\]

\[
\dot{Q}_2 = \varepsilon_2[\sigma(T_s^4 - 2T_{s1}^4 + \varepsilon_1T_{s2}^4 - 2T_{s2}^4)].
\]

We have introduced the temperatures \(T_i\) and the slab emissivities \(\varepsilon_i = \varepsilon(r_i)\) of each layer, where \(r_i\) is their mixing ratio \((i = 1, 2)\). Let \(w_i\) be their large-scale velocities. The interactive surface heat equation is still given by (5), but the radiative heating \(\dot{Q}_s\) is now given by equation (20). The boundary layer quasi-equilibrium can be written:

\[
F_1 + F_{b1} + F_m = F_s.
\]

The dynamics of water vapor in our model are analogous to equation (8), but complicated by the presence of a mid-tropospheric MSE flux and the vertical structure of the large-scale vertical velocity. In appendix B.1, we prove that they are approximately governed by the following equations:

\[
\frac{L_v\Delta p}{2g} \frac{\partial \alpha_r}{\partial t} = \left(1 - \varepsilon_p - \gamma\right)F_s + \left(1 - \gamma - \frac{\Delta h}{S_1}\right)\dot{Q}_1,
\]

\[
\frac{L_v\Delta p}{2g} \frac{\partial \alpha_r}{\partial t} = \alpha_\gamma F_s + \frac{\gamma \Delta h}{S_1} \dot{Q}_1 + \left(1 - \frac{\Delta h}{S_2}\right)\dot{Q}_2.
\]

\(\varepsilon_p\) is the precipitation efficiency, which is determined by microphysical processes. \(\Delta h \equiv h_s - h_m\) is the MSE contrast between the boundary layer and the mid-troposphere which generates the deep convection. \(S_1\) and \(S_2\) are respectively the lower and upper tropospheric DSE contrasts, defined by:

\[
S_1 \equiv \frac{\varepsilon_m - \varepsilon_b}{\varepsilon_b},
\]

\[
S_2 \equiv \frac{\varepsilon_b - \varepsilon_f}{\varepsilon_f}.
\]
\[ S_2 \overset{\text{def}}{=} \overset{\text{m}}{\pi} - \overset{\text{m}}{\pi}. \]  

\[ s_b, s_m, \text{and} \ s_t \text{are respectively the DSE of the boundary layer, the mid-troposphere, and the tropopause.} \ \gamma, \text{mathematically defined by equation (92), is the ratio of the upper to the lower tropospheric net convective updraft in RCE. Its variations are depicted on figure 8. The variables} (\alpha, \gamma, \epsilon, S_1, S_2) \text{are constant for a given RCE; they can be found in the table of parameters of the two-layer model.} \ \text{RCE is defined as the steady state of equations (5), (24) and (25) in the absence of large-scale vertical velocities (}\overset{\text{m}}{\pi} = \overset{\text{m}}{\pi} = 0). \ \text{From the full computation of RCE presented in appendix B.2, it can be proven that equation (9) still holds for the two-layer model: In RCE, the gross moist stability is zero (}\overset{\text{m}}{\pi} = 0) \text{and the surface convective flux balances the surface radiative heating, which is equal to the total radiative cooling of the atmosphere:} \]

\[ \overset{\text{m}}{\pi} = \overset{\text{m}}{\pi} = -\overset{\text{m}}{\pi} - \overset{\text{m}}{\pi}. \]  

\[ 3.2. \text{ Atmospheric radiative-convective instability for fixed surface temperature} \]

\begin{align*}
\text{Allowing for an atmospheric vertical structure gives the possibility of an atmospheric radiative-convective instability, which is studied in (Emanuel et al. 2014). However, previous studies neglected the vertical structure of convective updrafts, which is not valid if the vertical velocity is allowed to have a vertical structure. To isolate the atmospheric radiative-convective instability from the air-sea one studied in section 2, we do not allow the air-sea instability to happen by fixing the surface temperature to its RCE value} \overset{\text{m}}{\pi}. \ \text{We then study the stability of RCE when the lower and upper tropospheric mixing ratios} r_1 \text{and} r_2 \text{are perturbed from their equilibrium values, and how this stability depends on the RCE surface temperature} \overset{\text{m}}{\pi}. \\
\]

\[ 3.2.1. \text{ Linear analysis} \]

\begin{align*}
\text{We perform the linear analysis of the fixed surface temperature two layer model of the atmosphere by differentiating equations (24) and (25) with respect to the lower and upper tropospheric water vapor mixing ratios} r_1 \text{and} r_2: \\
\frac{\partial}{\partial r} \left( \begin{array}{c}
\overset{\text{m}}{\pi} \\
\overset{\text{m}}{\pi}
\end{array} \right) = J \left( \begin{array}{c}
\overset{\text{m}}{\pi} \\
\overset{\text{m}}{\pi}
\end{array} \right). \quad (29)
\]

\[ \text{Once again, we have denoted perturbations from RCE with primes, and the coefficients of the Jacobian} J \text{are given by:} \\
\text{J}_{11} = \frac{2g}{L_v \Delta p} \left( \frac{\partial Q_1}{\partial r_1} + \mathcal{L} \right), \quad (30)
\]

\[ J_{12} = \frac{2g}{L_v \Delta p} \left( \frac{\partial Q_1}{\partial r_2} + \mathcal{L} \right), \quad (31)
\]

\[ J_{21} = \frac{2g}{L_v \Delta p} \left( \frac{\partial Q_1}{\partial r_1} + 1 - \epsilon \frac{\partial Q_2}{\partial r_1} \right), \quad (32)
\]

\[ J_{22} = \frac{2g}{L_v \Delta p} \left( \frac{\partial Q_1}{\partial r_2} + 1 - \epsilon \frac{\partial Q_2}{\partial r_2} \right). \quad (33)
\]

\[ \text{We have defined:} \\
\Gamma = \frac{1}{1 - \epsilon}, \quad (34)
\]

\[ \lambda = -\frac{1}{1 - \epsilon} \frac{\partial Q_1}{\partial r_1}, \quad (35)
\]

\[ \text{which is defined in appendix B.2 by equation (98).} \ \mathcal{L} \text{represents the fact that changing the mid-tropospheric moisture can increase/decrease the MSE contrast between the boundary layer and the mid-troposphere, which enhances/inhibits the deep convection. The linear stability of the system can be determined by computing the eigenvalues of the Jacobian, which sum is given by (15) and product by (16). The sign of the determinant, given by (99), is governed by the value of} \ \gamma. \ \text{RCE is always linearly unstable in our model. The growth rate, defined as} \text{max}(R \lambda_1, R \lambda_2) \text{and depicted on figure 9, is maximal for low and high RCE surface temperatures, with a minimum when} \ \gamma \text{is equal amplitude. The system only oscillates for low RCE surface temperatures.} \\
\]

\[ 3.2.2. \text{ Physical mechanisms for high fixed surface temperature} \]

\[ \text{We aim at explaining the basic mechanism that reinforces an initially moist atmospheric perturbation, when the fixed surface temperature is high. For high temperatures, the lower troposphere is extremely moist and the upper troposphere is relatively moist. Because the slab emissivity of the atmosphere is concave (cf equation (19) and figure 5), it is almost constant for the very moist lower troposphere, whereas it still varies significantly with moisture for the less moist upper troposphere:} \]

\[ \frac{\partial \varepsilon}{\partial r_1} \ll \frac{\partial \varepsilon}{\partial r_2}. \quad (36)
\]

\[ \text{From equations (21) and (22), it means that the radiative heating will mostly vary following the upper tropospheric moisture} \]
variations:
\[
|\frac{\partial (\dot{Q}_1, \dot{Q}_2)}{\partial r_1}| \ll |\frac{\partial (\dot{Q}_1, \dot{Q}_2)}{\partial r_2}|.
\]

(37)

Finally, our numerical simulations of RCE show that a given upper tropospheric moist perturbation \( \delta r_2 > 0 \) warms the lower troposphere more than it cools the lower troposphere:
\[
\frac{\partial \dot{Q}_1}{\partial r_2} > -\frac{\partial \dot{Q}_2}{\partial r_2} > 0.
\]

(38)

Following figure 10, we now qualitatively explain the reinforcement of an upper tropospheric moisture perturbation \( \delta r_2 > 0 \). We do not taking into account the effects of perturbing the mid-tropospheric moisture, as they are not the primary instability mechanism for high temperatures. According to equation (38), the upper tropospheric moisture perturbation increases the radiative cooling of the upper troposphere, but it increases the radiative heating of the lower troposphere even more. Thus, not only does the lower troposphere become warmer and moister (generating a positive lower tropospheric moisture perturbation \( \delta r_1 > \delta r_2 \)), but a positive mid-tropospheric vertical velocity is also generated, advecting water vapor from the lower to the upper troposphere. This reinforces the upper tropospheric moisture perturbation, which explains the instability.

3.2.3. Physical mechanisms for low fixed surface temperature

For low fixed surface temperature, the previously described mechanism also applies, with the subtlety that lower tropospheric moisture perturbations now also affect the radiative heatings. However, the primary source of instability for low surface temperatures in the linear equations is \( \mathcal{C} > 0 \), introduced in equation (35). The fact that a moist perturbation in our system decreases the MSE contrast \( (h_b - h_m) \) which eventually amplifies the perturbation can be understood by studying the convective fluxes. Because of boundary layer quasi-equilibrium, we do not have to consider the variation of the fluxes \( (F_1, F_{b1}, F_{h1}) \) between the boundary layer and the lower troposphere, and focus on \( (F_m, F_2, F_{b2}) \). Our numerical simulation of RCE proves that for low temperatures:
\[
\frac{\partial F_m}{\partial r_1} > 0 \gg \frac{\partial (F_2 + F_{b2})}{\partial r_1} \sim 0 \quad i = 1, 2.
\]

(39)

Consequently, the reinforcement of the positive gross moist instability \( (F_m > 0 \text{ for } ) \) is the main trigger for instability when the surface temperature is low.

3.3. Effect of an interactive surface

We now allow the surface temperature \( T_s \) to vary in time, as we did in section 2. We expect both radiative-convective instabilities to be present: the air-sea and the air-air one. The goal is to study the nature of the full instability as a function of the RCE surface temperature \( T_s \) and the dimensionless surface heat capacity \( \mathcal{C} \). By differentiating equations (5), (24) and (25) with respect to the surface temperature \( T_s \) and the mixing ratios \( (r_1, r_2) \), we can...
obtain the linear version of our dynamical system:

\[
\frac{\partial}{\partial t} \begin{pmatrix} r_1' \\ r_2' \\ T_1' \\ T_2' \end{pmatrix} = J \begin{pmatrix} r_1' \\ r_2' \\ T_1' \\ T_2' \end{pmatrix}.
\]  

(40)

where \( J \) is the Jacobian matrix.

The coefficients of the Jacobian \( J \) are given in the appendix B.3. Its eigenvalues \((\lambda_1, \lambda_2, \lambda_3)\) can now be complex, but only their real part determine the linear stability of the system: the system will be unstable if and only if its growth rate \( \max(\Re \lambda_1, \Re \lambda_2, \Re \lambda_3) \), depicted on figure 11, is positive. The system is always linearly unstable. Furthermore, the heat capacity of the surface only significantly affects the growth rate of the instability for low RCE surface temperatures, consistently with the fact that the air-sea instability described in section 2 is most intense for low temperatures.

4. Single-column simulations

4.1. Method

We use the MIT SCM to examine the physics of the instability and its dry bias. The model is based on (Renno et al. 1994) and has been updated with the convection scheme of (Emanuel and Zivkovic-Rothman 1999) and the cloud scheme of (Bony and Emanuel 2001). The model has 46 levels in the vertical (the tropospheric vertical resolution is 25hPa) and a time step of 5min. The radiation is chosen to interact with water vapor, but the insolation is fixed to a constant value for a given simulation, and the clouds are deactivated for the sake of simplicity. The surface is interactive and turbulent enthalpy fluxes are computed using bulk aerodynamic formulas with an exchange coefficient of 1.2\(10^{-3}\) and a constant background wind speed of 5\(\text{ms}^{-1}\); its albedo is fixed at 0.20. Its heat capacity is equal to \( C_s = c_l \Delta Z_s \), where \( \Delta Z_s \) is the prescribed mixed layer depth for infrared radiation and \( c_l \approx 4.2\text{J.m}^{-2}.\text{K}^{-1} \) the volumetric heat capacity of liquid water. The greenhouse gas profile is fixed, except for the water vapor \( H_2O \) concentration, which is interactive: we prescribe a \( CO_2 \) concentration of 360.0ppm and a \( CH_4 \) concentration of 1.72ppm. Our experiments comprise two steps:

1. We run the MIT SCM to RCE. A small heat capacity is chosen (eg \( \Delta Z_s = 10\text{cm} \)) to make the adjustment timescale to RCE as small as possible (Cronin and Emanuel 2013). We always check that the system is in statistically steady state during the 50 last days of the simulation (which can be done by looking at the variables which are the slowest to adjust, such as precipitation and evaporation).

2. Each simulation is then reinitialized, starting from the RCE state. We introduce a uniform perturbation in water vapor mixing ratio to the profile (\( \pm 20\% \)) or no perturbation (\( +0\% \)). We then run the model in "WTG mode", ie the temperature is held fixed at and above 850hPa, and the vertical velocities (which can interact with the water vapor profile) are computed at and above 850hPa. This methodology has already been used in (Sobel and Bretherton 2000; Emanuel et al. 2014), and the goal here is to test the effect of allowing the surface to be interactive, as well as exploring a wider range of the parameter space \((T_s, C_s)\): we explore 20 surface temperatures ranging from 3°C to 41°C and 6 values of mixed layer depth, ranging from 10cm to 50m. We also introduce a background noise of 5% to the water vapor mixing ratio, in order for the system to not stay on one of the numerous limit cycles of each simulation.

4.2. Results

4.2.1. Effect of the interactive surface

Although (Emanuel et al. 2014) had neglected the direct effects of perturbing water vapor on convective heating, the radiative mechanisms of the atmospheric radiative-convective instability for fixed surface temperature are described thoroughly: As the surface temperature increases, the lower troposphere becomes opaque and the effect of the upper tropospheric moisture on the lower tropospheric radiative heating becomes big enough to amplify the water vapor mixing ratio perturbations. We ask ourselves how allowing the surface to be interactive changes the occurrence of the instability. In that purpose, we conduct the same experiments with a fixed surface temperature, and compare the range of RCE surface temperatures for which the instability occur, following the method described in the next paragraph. The results presented on figure 12 show that the instability occurs for a wider range of surface temperatures if the surface is allowed to be interactive, independently of the heat capacity of the surface (as long it is not extremely high). This is consistent with the fact that a fixed surface temperature inhibits the air-sea radiative-convective instability, which reduces the growth rate of the instability. However, RCE is always linearly unstable to coupled perturbations in our model, even for very low temperatures. This may be due to the fact that we have neglected ice physics (incorporated in the MIT SCM), which could inhibit the instability for low temperatures by limiting the water vapor content of the atmosphere. Because of the intrinsic variability of the system, the nature of the final state is slightly random: typically 10-20% of the experiments will not go unstable despite the fact that their surface temperature is in the unstable range.

<table>
<thead>
<tr>
<th>Fixed surface temperature: Minimal temperature at which the experiments go unstable</th>
<th>-20%</th>
<th>15°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>40°C</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>25°C</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Interactive surface: Minimal temperature at which the experiments go unstable</th>
<th>-20%</th>
<th>10°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>40°C</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>10°C</td>
<td></td>
</tr>
</tbody>
</table>

Figure 12. Minimal temperature at which the MIT SCM experiments go unstable for different experiment types.
from. The equilibrium moisture profile reflects the way it has been advected: the upwards vertical velocity in the (+20%) case makes the lower and upper troposphere anomalously moist; whereas the downwards vertical velocity in the (−20%) case dries out the upper troposphere and moistens the boundary layer as well as the lower troposphere. The asymmetry of the total heating comes from the fact that in the (+20%) case, the large upwards vertical velocity makes the high moisture content of the upper troposphere rains off, producing a significant amount of latent heat. The asymmetry of the vertical velocity itself could come from the fact that downwards velocities are radiatively bounded by the clear sky DSE balance (54):

$$w_{ls} = \frac{\varepsilon_{fc}}{\rho_a} \left[ \frac{\dot{Q}_a}{R} \right] \frac{\dot{Q}_a}{\rho_a (s_b - s_a)} \geq 0,$$

whereas there are no equivalent upper bounds for the upwards velocities. Furthermore, in the dry zones of mean descent, there is a lower bound for the precipitation (0mm/day or equivalently a 100% decrease compared to the RCE value), which is often reached in the (−20%) experiments. However, the relative upper bound for precipitation is much higher, and the (+20%) experiments often show increase in precipitation larger than 100%, compared to the RCE value.

5. Conclusion

We have shown that under the WTG approximation, the presence of an interactive surface allowed a new instability to develop: the air-sea radiative-convective instability. It relies on the mutual reinforcement of water vapor and surface temperature perturbations through the free-tropospheric radiative cooling, the surface radiative heating and the surface MSE flux. It occurs for all RCE surface temperatures and surface capacities, but its growth rate is maximal for low temperatures and low surface heat capacities. The occurrence of this instability relies on the variation of the longwave emissivity with moisture, which means the radiation has to be interactive for it to operate. Furthermore, lateral advection of water vapor and oceanic advection of heat must be negligible for this instability to occur.

For low RCE surface temperatures (or equivalently low insolation), this instability couples with the purely atmospheric radiative-convective instability, where the upper and lower water vapor mixing ratios mutually reinforce through the atmospheric radiative coolings and the mid-tropospheric MSE flux. This leads to a full radiative-convective instability with larger growth rates. Consistently with previous studies, negative moisture perturbations give rise to a dry and cool final state with large-scale descent, whereas positive moisture perturbations induce a moist and warm state, with large-scale ascent.

We have established that this radiative-convective instability has a dry bias:

- In the sense that the dry instability grows faster than the moist instability, which is related to the concavity of the emissivity as a function of the water vapor mixing ratio.
- In the sense that the final moist state has higher absolute vertical velocities than the final dry state, which explains why it occupies a smaller area through mass conservation. This asymmetry is believed to be caused by the lower bounds on the clear-sky vertical velocity and the precipitation, that can be reached in the dry state.

This study has been confined to a single column subjected to a vertical large-scale circulation, but it can be generalized to a given domain by putting several columns next to each other. It would be interesting to study the dry bias of this instability when
the system is subjected to $f$–plane or $\beta$–plane dynamics, in order to apprehend the climatological consequences of weather disturbances on water vapor. It is important to note that the theoretical part of this study is highly idealized, which presents several key limitations:

- The model does not hold for long times, since the radiative effects of clouds play a big role in maintaining/dissipating the dry and the moist zones after RCE has gone unstable.
- The model does not hold for low surface temperatures. Indeed, we have neglected ice physics, and we have made the WTG approximation which only applies where the Coriolis parameter is small, which on Earth is where the temperatures are high.

A. Air-sea radiative-convective instability - Supplementary material

A.1. One layer model of the tropical atmosphere

The full energy balance of the system is separated in three parts: convective in the cumulus region (brown arrows), convective in the clear-sky region and radiative (red arrows). This appendix is meant to complete the equations of the one layer model presented in section 2.1.

A.1. Parametrization of the convective fluxes

A bulk aerodynamic formula is used to compute the upwards surface enthalpy flux $F_s$:

$$ F_s = \rho_s C_E U_s (h_s - h_b). $$

$\rho_s$ is the near-surface air density, $U_s$ the near-surface wind speed, $C_E$ the dimensionless enthalpy exchange coefficient (all are assigned a fixed value listed in table 2), $h_s$ the surface MSE and $h_b$ the MSE at the top of the boundary layer. We now separate the column in two parts: the area where cumulus convection occurs, occupying a fractional area $S_{\text{cum}}$; and the clear-sky area, occupying a fractional area $1 - S_{\text{cum}}$. The large scale vertical velocity $w_{ls}$ of the air column is defined as the area-weighted average of the vertical velocity in each part:

$$ w_{ls} \overset{\text{def}}{=} \frac{S_{\text{cum}} w_{\text{cum}} + (1 - S_{\text{cum}}) w_{\text{clear-sky}}}{M_u}, $$

where we have defined the deep updraft mass flux $M_u$ and the subsidence mass flux $M_s$. Following (Yano and Emanuel 1991), we take into account the fact that a significant part of the convection is not deep and does not rain out of the atmosphere (e.g. shallow clouds, non-precipitating clouds, or rainy downdrafts with a return flow in the free troposphere). The latter type of convection has equal updrafts and downdrafts mass fluxes, of amplitude $M_d$,
that we parametrize as proportional to the total updraft mass flux \( M = M_a + M_c \):

\[
M_d = (1 - \varepsilon_p)M. \tag{44}
\]

where the precipitation efficiency \( \varepsilon_p \) is a prescribed constant determined by microphysics (typically between 0.7 and 1). The MSE flux from the boundary layer to the free troposphere comes from the fact that cloudy downdrafts make the boundary layer loose entropy to the free troposphere; it can be parametrized by:

\[
F_a \approx \rho_a M_d(b_h - h_a) = \rho_a (1 - \varepsilon_p)M(b_h - h_a), \tag{45}
\]

where we have used the assumption (44). \( \rho_a \) and \( h_a \) are the averaged free-tropospheric air density and MSE. The net subsidence in the boundary layer within the clear-sky environment is given by:

\[
F_b \approx -\rho_a M_b(h_b - h_a) = \rho_a (\varepsilon_p M - w_{ua})(h_b - h_a), \tag{46}
\]

where we have used the definition (43).

A.1.2. Parametrization of the emissivity

Writing the slab emissivity \( \varepsilon \) as a function of the water vapor mixing ratio \( r_a \) (or equivalently as a function of the water vapor partial pressure \( e \), approximately proportional to \( r_a \)), is a good way to circumvent the resolution of the full radiative transfer equation. Early empirical formulas, such as (Brunt 1936), suggest that the emissivity vary like the square root of \( r_a \), offset by a constant. Early derivable formulas, such as (Brutsaert 1975), fit power laws to the emissivity profile \( \varepsilon(r_a) \). We make the second choice and fit a power law to the MIT SCM results, using the approximate method depicted on figure 15. This method requires the definition of:

- The pressure at the top of the atmosphere \( p_{\text{TOA}} \):

\[
X^{\text{trop}}_{\text{def}} = \frac{1}{(p_h - p_t)} \int_{p_t}^{p_h} X dp. \tag{47}
\]

- An approximate mass-weighted free-tropospheric average:

\[
X^{\text{trop}}_{\text{def}} = \frac{1}{(p_t - p_{\text{TOA}})} \int_{p_{\text{TOA}}}^{p_t} X dp. \tag{48}
\]

It does not require any knowledge of the radiative profile, and is based on \( F_{\text{TOA},\text{LW}} \), the outgoing longwave flux at the top of the atmosphere. We use the fact that the convective fluxes can be approximately neglected above the tropopause. The longwave energy balance of the "stratospheric and above" in RCE can be written using the previous notations:

\[
Q^{\text{strat}}_{\text{rad}} \approx \sigma [\varepsilon T^{\text{trop}}_{\text{rad}} + (1 - \varepsilon)T^4_{s}] + F_{\text{TOA},\text{LW}}. \tag{49}
\]

The slab emissivity as a function of the free tropospheric mixing ratio is then given by:

\[
\varepsilon(r_a^{\text{trop}}) \approx \frac{T^4_{s} + \sigma^{-1}(\varepsilon^{\text{strat}}_{\text{rad}} - F_{\text{TOA},\text{LW}})}{T^4_{s} - (T^{4}_{s})}. \tag{50}
\]

On figure 15, we can see that the "slab emissivity approximation" does not hold well when the air column is too moist, but does reasonably well otherwise:

\[
\varepsilon_{\text{fit}}(q_a) \approx \begin{cases} 0.72q_a \text{[g.kg}^{-1}] & 0 \leq q_a < 9 \text{g/kg} \\ 1 & q_a \geq 9 \text{g/kg} \end{cases}, \tag{51}
\]

where we have defined the free tropospheric specific humidity as:

\[
q_a = \frac{r_a}{1 + r_a}. \tag{52}
\]

A.1.3. Full equations of the model

Equations (5), (6) and (8) are derived in section 2.1.3. We can express the MSE fluxes in the cumulus region as functions of the updraft mass fluxes and the large-scale vertical velocity by combining equation (6) to the parameterizations (45) and (46):

\[
F_a = \rho_a (b_h - h_a)(M - w_{ua}). \tag{53}
\]

Following (Arakawa and Schubert 1974), we assume that all the condensation and evaporation occurs within the clouds, which occupy a small fractional area. As a consequence, the convective heating in the free troposphere is given to a good approximation by its clear-sky value \( M_a(s_b - s_a) \). Furthermore, according to definition (1), the DSE is steady under the WTG approximation. The vertically integral of the DSE budget in the free-troposphere can thus be written

\[
\rho_a w_{ua}(s_b - s_a) = \dot{Q}_a + \rho_a \varepsilon_p M(s_b - s_a). \tag{54}
\]

A.1.4. Full equations of RCE

RCE is defined as the steady state of equations (5), (8), (53) and (54) \((\frac{\partial}{\partial t} = 0)\) with no large-scale vertical velocity \((\vec{\sigma}_{\text{LS}} = 0)\). Combining equation (9) with equations (53) and (54) in RCE, we obtain a relation between the MSE and the DSE contrasts in RCE:

\[
\overline{h_b} - \overline{h_a} = \varepsilon_p(\overline{s_b} - \overline{s_a}). \tag{55}
\]

Equations (9) and (55) are insufficient to numerically compute the RCE temperatures and pressures in our model, and we need to add several assumptions:

- Strict quasi-equilibrium is the assumption that the temperature profile of the atmosphere follows a moist adiabat. It is based on meteorological observations of free tropospheric temperatures profiles within convective regions, and is well verified in the 800hPa – 200hPa layer (Holloway and Neelin 2007). In the one layer model, it implies that the saturated MSE \( h^* \), defined by replacing the mixing ratio \( r \) by its saturation value \( r^* \) in definition (2), is approximately conserved in the vertical \((h_b^* = h_a^*)\).
- We assume that the surface is saturated, which means that the mixing ratio \( r \) at the surface is equal to its saturation value \( r^*(T_s, p_s) \).
- The tropopause is assumed to be a thin blackbody layer of temperature \( T_t \). It is transparent enough not to change the radiative budget, and the energy balance at the top of the atmosphere yields: \( T_t = 2^{\frac{3}{4}}T_a \).
- The free-tropospheric layer’s pressure level \( p_a \) is equidistant from the surface and the tropopause \((p_s - p_a = p_a - p_t)\).
- In RCE, the boundary layer’s altitude \( z_b \) is prescribed as well as the temperature’s difference between the surface and the boundary layer \((T_a - T_b)\).

A.2. Instability to water vapor and surface temperature perturbations

A.2.1. Dynamics of the one layer model

As soon as we perturb the free-tropospheric mixing ratio \( r_a \) and the surface temperature \( T_s \), RCE does not hold anymore. Equations (5), (8) form a nonlinear dynamical system, that we need to solve in order to compute the evolution of \( r_a \) and \( T_s \). In that purpose, additional physical constraints are needed. Because we have made the WTG approximation, the temperatures of the
atmosphere and the boundary layer are steady, which means that the DSE contrast \((s_h - s_a)\) will always be equal to its initial value, that we call \(S\). In contrast, the MSE of the boundary layer \(h_b\) and of the free troposphere \(h_a\) will vary in time because of moisture variations. However, in order to compute the MSE flux from the boundary layer to the free troposphere \(F_a\) \((45)\), we neglect the variations of the MSE contrast \((h_b - h_a)\) in time, and assume that it remains equal to its RCE value: \(T_p S\), according to equation \((55)\). This is a strong assumption that will most likely not hold for long times, but results from the MIT SCM show that it is significantly more accurate than assuming that the boundary layer MSE \(h_b\) is steady \((\text{Emanuel et al. 2014})\). Finally, we assume that the surface MSE flux \(F_s\) \((42)\) evolves linearly with the surface temperature, according to its unsaturated thermal inertia:

\[
\frac{\partial F_s}{\partial T_s} \approx \rho_s C_E U_s \frac{\partial (h_s - h_b)}{\partial (T_s - T_b)} = \rho_s C_E U_s c_p, \tag{56}
\]

where we have used definition \((2)\) to write the second equality. Experiments with the MIT SCM show that equation \((56)\) estimates \(\frac{\partial F_s}{\partial T_s}\) fairly accurately. In contrast, assuming that the surface stays saturated greatly overestimates the latter quantity.

### A.2.2. Linear analysis

Combining equations \((11), (4)\), and remembering that the temperatures are fixed in our system:

\[
J_{11} = \frac{g}{L_v \Delta p} \frac{Q_a}{\varepsilon} \frac{\partial \varepsilon}{\partial r_a}. \tag{57}
\]

Combining equations \((12), (4)\) and \((42)\):

\[
J_{12} = 4 \sigma \frac{T_s^4}{L_v \Delta p} (\varepsilon + \tilde{F}). \tag{58}
\]

To write \((58)\), we have introduced the following dimensionless number:

\[
\tilde{F} \equiv \left( \frac{\partial F_s}{\partial T_s} \right) \left( \frac{\partial Q_a}{\partial T_a} \right)^{-1} = \frac{\rho_s C_E U_s c_p}{4 \sigma T_s^4}, \tag{59}
\]

and combined the approximation \((56)\) with equation \((3)\). In this model, for a range of RCE surface temperature between 10Cand 40C, we find that \(\tilde{F}\) is order 1 and larger than 1. Combining equations \((13)\) and \((3)\):

\[
J_{21} = \frac{\sigma}{C_s} \frac{T_s^4}{T_a} \frac{\partial \varepsilon}{\partial r_a}. \tag{60}
\]

Finally, combining equations \((14), (42), (3)\) and \((59)\):

\[
J_{22} = -4 \sigma \frac{T_s^3}{C_s} (1 + \tilde{F}). \tag{61}
\]

The determinant \((16)\) is then given by:

\[
-4 \sigma \frac{T_s^3}{C_s} \left( \frac{\partial Q_a}{\partial T_a} \right) \left[ \frac{T_s^3}{T_s^3} (1 + \tilde{F}) - T_a^3 (2 - \varepsilon + \tilde{F}) \right]. \tag{62}
\]

The sign of the determinant \((62)\) must be determined numerically: we find that it is always negative for RCE surface temperatures between 10Cand 40C. The growth rate \((17)\) is given by:

\[
2 \sigma \frac{T_s^3 r\ref T_s}{r\ref T_s \Delta p} \left[ \left( \frac{1 + \tilde{F}}{\tilde{C}_s} - Q \right)^2 - \frac{4QT_s^2}{T_s^3} \pi + \tilde{F} - \left( \frac{1 + \tilde{F}}{\tilde{C}_s} + Q \right) \right]. \tag{63}
\]

In order to write equation \((63)\), we have introduced the dimensionless surface heat capacity defined by \((18)\) and defined:

\[
\tilde{Q} \equiv r\ref \left( \frac{\partial Q_a}{\partial T_a} \right) \left( \frac{\partial Q_s}{\partial T_s} \right)^{-1} = -\frac{r\ref Q_a}{4 \sigma T_s^3 r\ref \tilde{C}} \frac{\partial \varepsilon}{\partial r_a}. \tag{64}
\]

With the reference mixing ratio: \(r\ref = 1\%\), \(\tilde{Q}\) is order 10. The asymptotic growth rate \(\lambda_1\) can be found by asymptotically expanding \((63)\) for \(\tilde{C} \ll 1\), yielding:

\[
\lambda_1 \approx \frac{g \pi}{L_v \Delta p} \frac{\partial \ln \varepsilon}{\partial r_a} (T_s^3 + (1 + \tilde{F}) [\varepsilon + \tilde{F} - 2]) \frac{T_s^3}{T_s^3}. \tag{65}
\]

The right part of equation \((65)\) increases with RCE surface temperature. However, because of the concavity of the slab emissivity \((19)\), the middle part of equation \((65)\) decreases very sharply with RCE surface temperature, and reaches zero as soon as the emissivity saturates \((\sigma r \gtrsim 9 g kg^{-1} \text{ in our model})\). In total their product \(\lambda_1\) decreases with RCE surface temperature and reaches zero when the emissivity saturates.
A.2.3. Expansion to second order

In order to quantitatively describe the autonomous part of the dry bias of this instability, we can expand the dynamical system formed by equations (5) and (8) to second order. We define the vector \( \mathbf{z} = \left[ r_a \ T_s \right]^T \) so that the expansion can be written:

\[
\frac{\partial x_i}{\partial t} = \sum_{j=1}^{2} J_{ij} x_j + \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} x_j H_{ijk} x_k + O(x_i^3), \tag{66}
\]

where \( i = 1, 2 \). We have used the Jacobian \( J \) which coefficients are computed in section B.2, and introduced the Hessian \( H \), which coefficients can be computed by differentiating twice equations (5) and (8) with respect to \( r_a \) and/or \( T_s \):

\[
H_{111} = \frac{g Q_a}{L_v \Delta p} \frac{\partial^2 \varepsilon}{\partial r_a^2} (> 0), \tag{67}
\]

\[
H_{112} = H_{211} = \frac{4 \sigma T_a^2}{L_v \Delta p} \frac{\partial \varepsilon}{\partial r_a} (> 0), \tag{68}
\]

\[
H_{212} = \frac{12 \sigma T_a^2}{L_v \Delta p} (> 0), \tag{69}
\]

\[
H_{121} = \frac{4 \sigma T_a^2}{C_a \Delta p} \frac{\partial^2 \varepsilon}{\partial r_a^2} (< 0), \tag{70}
\]

\[
H_{122} = H_{221} = 0, \tag{71}
\]

\[
H_{222} = -\frac{12 \sigma T_a^2}{C_a} (< 0). \tag{72}
\]

For a given perturbation, the second term of the expansion \( 66 \) is thus:

\[
\frac{1}{2} \left[ H_{111}(r_a')^2 + (H_{112} + H_{211})(r_a'T_s') + H_{212}(T_s')^2 \right] \frac{\partial^2 \varepsilon}{\partial r_a^2} > 0, \tag{73}
\]

\[
\frac{1}{2} \left[ H_{121}(r_a')^2 + H_{222}(T_s')^2 \right] \frac{\partial^2 \varepsilon}{\partial r_a^2} < 0. \tag{74}
\]

The fact that the two components of the stability are of opposite sign in equation \( 73 \) can explain part of the drying for short times and moderate surface temperatures. We note the importance of the concavity of the emissivity profile \( \frac{\partial^2 \varepsilon}{\partial r_a^2} \) for this drying effect to occur.

B. Full radiative convective instability - Supplementary material

B.1. Two layer model of the tropical atmosphere

B.1.1. Parametrization of the convection

The two layer model of the atmosphere relies on the large-scale vertical velocity of the lower troposphere \( w_1 \) and of the upper troposphere \( w_2 \), respectively defined as:

\[
w_1 \equiv M_{u1} + M_{b1}, \tag{74}
\]

\[
w_2 \equiv M_{u2} + M_{b2}. \tag{75}
\]

We have defined the upper and lower tropospheric deep updraft mass fluxes \( M_{u1} \) and subsidence mass fluxes \( M_{b1} \) \((i = 1, 2)\). Again, we take into account that a fraction of the convection is not deep and does not rain out of the atmosphere. We divide the total lower tropospheric mass flux \( M_1 \) in two proportional parts using the lower tropospheric precipitation efficiency \( \epsilon_p \):

\[
M_1 = \epsilon_p M_1 + (1 - \epsilon_p) M_1, \tag{76}
\]

and assume that the second part \( M_{b1} \) represents the lower tropospheric non-deep convection, with equal updrafts and downdrafts. We repeat this approximation in the upper troposphere, where \( M_2 \) is defined to be the net convective updraft:

\[
M_2 = \epsilon_p M_2 + (1 - \epsilon_p) M_2. \tag{77}
\]

Our equations make it easy to give a vertical structure to the precipitation efficiency, but we neglect it here for the sake of simplicity.

The cumulus MSE fluxes in our model (analogous to \( F_a \) given by \( 45 \)) can be approximated by:

\[
F_1 \approx \rho M_{d1}(h_b - h_m) = \rho(1 - \epsilon_p) M_1(h_b - h_m), \tag{78}
\]

\[
F_2 \approx \rho M_{d2}(h_b - h_m) = \rho(1 - \epsilon_p) M_2(h_b - h_m). \tag{79}
\]

We have used the two previous equations \( 76, 77 \) and introduced \( \rho \), the typical density of the atmosphere. We have also defined the mid-tropospheric MSE as the average of the lower and the upper tropospheric MSE:

\[
h_m \equiv \frac{L_v r_1 + r_2}{2} \tag{80}
\]

where \( s_m \) is the mid-tropospheric DSE. The subsidence MSE fluxes (analogous to \( F_t \) given by \( 46 \)) can be approximated by:

\[
F_{b1} \approx -\rho M_{b1}(h_b - h_m) = \rho(\epsilon_p M_1 - w_1)(h_b - h_m), \tag{81}
\]

\[
F_{b2} \approx -\rho M_{b2}(h_b - h_m) = \rho(\epsilon_p M_2 - w_2)(h_b - h_m). \tag{82}
\]

Following \( 34 \), the gross moist stability is generally defined as:

\[
F_m \equiv \left( \frac{\partial h_m}{\partial z} \right)_{\text{trop}}, \tag{83}
\]

where we have used the mass-weighted tropospheric average defined in equation \( 47 \). In our discrete two-layer model, this term is approximated by:

\[
F_m \approx \rho [w_1(h_b - h_m) + w_2(h_m - h_t)] = \rho(w_1 - w_2)(h_b - h_m), \tag{84}
\]

where we have assumed a symmetric MSE profile with a mid-tropospheric minimum, such that:

\[
h_b - h_m = h_t - h_m. \tag{85}
\]

B.1.2. Full equations of the model

We start from equations \( 5 \) and \( 23 \). By combining equation \( 23 \) to the parametrization of the lower tropospheric cumulus flux \( 78 \) and subsidence flux \( 81 \), we can relate the surface flux to the lower tropospheric updraft and large-scale vertical velocity:

\[
\alpha F_s = \rho(h_b - h_m)(M_1 - w_1). \tag{86}
\]

The MSE budgets of the lower troposphere yields:

\[
\frac{\partial h_1}{\partial t} \approx \frac{2 g}{L_v} \frac{\partial T_G}{\partial t} \approx \frac{2 g}{L_v} (\bar{Q}_1 + F_s + F_h + F_{b1} + F_m - F_2 - F_{b2}). \tag{87}
\]

Using the boundary-layer quasi-equilibrium \( 23 \), one can simplify the evolution of the lower tropospheric mixing ratio:

\[
L_v \Delta p \frac{\partial r_1}{\partial t} = \bar{Q}_1 + F_s + F_m - F_2 - F_{b2}. \tag{88}
\]

Similarly, the evolution of the upper tropospheric mixing ratio is given by:

\[
L_v \Delta p \frac{\partial r_2}{\partial t} = \bar{Q}_2 + F_2 + F_{b2}. \tag{89}
\]
Finally, in order to write the DSE budget, we approximate again the convective heating in the lower and upper tropospheric regions by their clear-sky value, respectively \( M_{a1} \) and \( M_{a2} \), where \( S_1 \) and \( S_2 \) are the DSE contrasts defined in (26) and (27). The analogous of equation (54) in our model is now:

\[
\rho \omega_1 S_1 = \dot{Q}_1 + \rho c_p M_1 S_1, \tag{90}
\]

\[
\rho \omega_2 S_2 = \dot{Q}_2 + \rho c_p M_2 S_2. \tag{91}
\]

Combining equations (90), (91) and (86), it is possible to express the vertical motion variables \((M_1, M_2, \omega_1, \omega_2)\) as functions of the radiative heating and surface flux: \((\dot{Q}_1, \dot{Q}_2, F_s)\). From the combination of (88), (89), and the parametrization of the convective fluxes, we can express the dynamics of water vapor as a function of \((\dot{Q}_1, \dot{Q}_2, F_s)\), which leads to equations (24) and (25).

### B.1.3. Full equations of RCE

RCE is defined as the steady state of equations (5), (86), (88), (90) and (91) \( \frac{\partial h}{\partial t} = 0 \) with no large-scale vertical velocities \( (\omega = 0 = 0) \). Combining equations (90) and (91), we obtain a constraint on the ratio of the updrafts and downdrafts in RCE, that we call \( \gamma \):

\[
\gamma \overset{\text{def}}{=} \frac{M_2}{M_1} = \frac{S_1 \dot{Q}_2}{S_2 \dot{Q}_1}. \tag{92}
\]

Combining equations (23), (88) and (89), we obtain a constraint on the shallow convection fraction in RCE:

\[
\bar{\tau} = \frac{\dot{Q}_2}{\dot{Q}_1} = \frac{S_2}{S_1} = \frac{\ddot{Q}_1}{\dot{Q}_1 + \dot{Q}_2}. \tag{93}
\]

Finally, combining equations (89) and (91) in RCE yields:

\[
\bar{h}_b - \bar{h}_m = \epsilon_p S_2. \tag{94}
\]

Equations (28), (93), (92) and (94) do not entirely determine RCE in our two-layer model, and we need to add the assumptions listed at the end of Section A.1.4.

### B.1.4. Dynamics of the two layer model

RCE breaks as soon as we perturb the mixing ratios \( r_1 \) and \( r_2 \) or the surface temperature \( T_s \). We integrate equations (5), (24) and (25) in time. Because of the WTG approximations, the DSE \( s_b, s_m \) and \( s_t \) are steady, which means that the DSE contrasts are steady:

\[
s_m - s_b = S_1, \tag{95}
\]

\[
s_t - s_m = S_2. \tag{96}
\]

Furthermore, we assume convective neutrality between the boundary layer and the free troposphere, which means that the variations of the MSE contrast between the boundary layer and the mid-troposphere are directly related to the changes in the mixing ratios \( r_i \) (\( i = 1, 2 \)):

\[
\frac{\partial (h_b - h_m)}{\partial r_i} = -\frac{\partial h_m}{\partial r_i} = -\frac{L_v}{C_v} \tag{97}
\]

where we have used equation (80). We assume again that the tendency of the surface convective flux with temperature is given by (56). For the sake of simplicity, we assume that the shallow-convective fraction of the surface flux and the ratio of the upper and lower tropospheric updrafts stay at their RCE values for all times \( \alpha = \bar{\tau} \) and \( M_2 = \gamma M_1 \).

### B.2. Atmospheric radiative-convective instability with fixed surface temperature: Linear analysis

The linear analysis of equations (24) and (25) is complicated by the fact that the MSE contrast \((h_b - h_m)\) varies with the mixing ratios \( r_i \). It is convenient to define:

\[
\mathcal{L} \overset{\text{def}}{=} \frac{1 - \gamma_s}{1 - \epsilon_p} \frac{\partial (h_b - h_m)}{\partial r_i}, \tag{98}
\]

which leads to equation (35) if combined with equation (94). The product of the eigenvalues is given by the determinant of the Jacobian; it equals:

\[
\lambda_1 \lambda_2 = \left( 1 - \epsilon_p \right) \left( \sqrt{\frac{S_2 \gamma_p L_s}{S_1 \dot{Q}_1}} \right)^2 \left( \frac{\partial \dot{Q}_1}{\partial \dot{r}_1} - \frac{\partial \dot{Q}_1}{\partial \dot{r}_1} \right) + \left( \frac{\partial \dot{Q}_1}{\partial \dot{r}_1} \right)^2 + \left( \frac{\partial \dot{Q}_1}{\partial \dot{r}_1} \right)^2, \tag{99}
\]

\[
\lambda_1 \lambda_2 = \left( 1 - \epsilon_p \right) \left( \sqrt{\frac{S_2 \gamma_p L_s}{S_1 \dot{Q}_1}} \right)^2 \left( \frac{\partial \dot{Q}_1}{\partial \dot{r}_1} \right)^2 + \left( \frac{\partial \dot{Q}_1}{\partial \dot{r}_1} \right)^2, \tag{100}
\]

The monotony of each term with the RCE surface temperature \( T_s \) is indicated in equation (99) by up/down arrows, as well as the temperature at which each term changes sign. The first term is almost an order of magnitude larger than the second term, and is sufficient to determine the sign of the determinant in practice. Note that most quantities change sign when the RCE surface temperature is 25°C in our model, because \( \gamma \) reaches 1 at that temperature. For \( T_s < 25°C \), \( \lambda_1 \lambda_2 > 0 \) and both eigenvalues are complex with a positive real part; for \( T_s > 25°C \), \( \lambda_1 \lambda_2 < 0 \) and both eigenvalues are real and have opposite signs. The system is thus always linearly unstable but only oscillates for \( T_s < 25°C \).

### B.3. Effect of an interactive surface: Linear analysis

The first four coefficients of the Jacobian defined in (40) are equal to \( J_{11} \) given by (30), \( J_{12} \) given by equation (31), \( J_{21} \) given by equation (32) and \( J_{22} \) given by equation (33). The other coefficients are listed below:

\[
J_{13} = \frac{2 \gamma}{L_v \Delta p} \left[ (1 - \gamma_s) \frac{\partial F_s}{\partial T_s} + S_2 \gamma_p L_s \frac{\partial \dot{Q}_1}{\partial \dot{r}_1} \right], \tag{101}
\]

\[
J_{23} = \frac{2 \gamma}{L_v \Delta p} \left[ \gamma_s H_s \frac{\partial F_s}{\partial T_s} + \frac{S_2 \gamma_p L_s}{S_1} \frac{\partial \dot{Q}_1}{\partial \dot{r}_1} + (1 - \epsilon_p) \frac{\partial \dot{Q}_2}{\partial \dot{T}_s} \right], \tag{102}
\]

\[
J_{31} = \frac{\partial \dot{Q}_s}{\partial C_s \dot{r}_1}, \tag{103}
\]

\[
J_{32} = \frac{\partial \dot{Q}_s}{\partial C_s \dot{r}_2}, \tag{104}
\]

\[
J_{33} = \frac{\partial \dot{Q}_s}{\partial C_s \dot{T}_s}, \tag{105}
\]

Table 1. Acronyms of the paper

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Definition</th>
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<tr>
<td>DSE</td>
<td>Dry static energy</td>
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<td>MIT SCM</td>
<td>MIT single column model</td>
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<tr>
<td>MSE</td>
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<td>RCE</td>
<td>Radiative convective equilibrium</td>
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<td>WTG</td>
<td>Weak temperature gradient</td>
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Table 2. Parameters of the one layer model

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Table 3. Additional parameters for the two layer model

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