A correlated stochastic model for the large-scale advection, condensation and diffusion of water vapor

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The statistically steady distribution of water vapor and its characteristics are studied in the framework of the Ornstein-Uhlenbeck (OU) turbulent dispersion model. Particles are advected by a stochastic velocity field with a finite time correlation, and condense as soon as their moisture contents exceeds a local saturation value. We discretize the OU model to a finite number of velocity values, and find simple analytical solutions for the bimodal distribution of water vapor and the non diffusive moisture flux, which perfectly agree with the corresponding Monte-Carlo simulations. Furthermore, we show that these simple models produce results that approximate well the OU Monte-Carlo simulations, suggesting that they could be used as general tools to understand correlated stochastic processes that involve condensation.

Key Words: advection; condensation; diffusion; distribution; flux; saturation; specific humidity; stochastic

1. Introduction

The role of water vapor in Earth’s climate cannot be overstated. As the main greenhouse gas in the atmosphere, variations of its concentration in any part of the free troposphere have a direct radiative effect (Spencer and Braswell 1996), and can significantly affect climate sensitivity (Soden et al. 2005). For example, the simple presence of water vapor on Earth is responsible for a 50 – 100 W m⁻² drop in outgoing longwave radiation, and assuming that the atmosphere is saturated rather than resolving its distribution leads to errors as large as 20 W m⁻² (Pierrehumbert, Brogniez and Roca 2007). Because the variability of water vapor is closely tied to moist convection and the formation of clouds (Brown and Zhang 1997), it also has important indirect radiative effects, as well as effects on atmospheric dynamics, such as the strength of the Hadley cell and the extra-tropical baroclinic eddies (Schneider, O’Gorman and Levine 2010). It also plays a central meteorological role. Important weather systems such as cloud clusters or cyclones can be viewed as aggregates of water vapor and liquid water, and it is crucial to understand the interplay of the organization of convection and water vapor (Tompkins 2001; Muller and Held 2012; Wing and Emanuel 2014).

For high enough latitudes and long enough timescales, models that only rely on the advection and condensation of water vapor, namely the ”Advection-Condensation (AC) models”, can explain the large-scale distribution of water vapor to a reasonable approximation (Dessler and Sherwood 2000; Sherwood, Kursinski and Read 2006; Sherwood 1996; Pierrehumbert and Roca 1998). However, the AC models neglect microphysical processes such as the incomplete fallout of precipitation (Liu Fueglistaler and Haynes) and mixing with the surface (Hurley et al. 2012), which are important on short timescales. As a consequence, they are unable to resolve the microstructure of the water vapor distribution and have a dry bias. To understand the fundamentals of large-scale water vapor distribution, the stochastic version of the AC model was subsequently introduced (Pierrehumbert, Brogniez and Roca 2007): Assuming a one-dimensional Brownian motion for the air parcels and an exponentially decreasing saturation specific humidity profile, initial value problems were solved and the probability distribution function (PDF) of moisture was obtained numerically. O’Gorman and Schneider (O’Gorman and Schneider 2006; O’Gorman et al. 2011) added correlation to the model by assuming that the stochastic velocity field was governed by an Ornstein-Uhlenbeck (OU) process (Uhlenbeck and Ornstein 1930), switching from white noise to Gaussian colored noise. They were able to analytically compute the moisture flux and the condensation rate in the Ballistic and Brownian limits of the OU model, and computed the same quantities numerically in the general correlated case. Sukhatme and Young (Sukhatme and Young 2011) were able to analytically solve for the PDF of moisture, the moisture flux and the condensation rate in the general case, providing significant insight in the physics of the process. However, they had to return to a white noise process for the velocity field.

Here, we ask the question: What are the physics of the stochastic AC model in the case where the velocity process has a finite correlation time (typical time for a parcel to change direction)? Figure 1 depicts how a correlated process compares to its two limits: the Brownian case (zero correlation time τ) and the Ballistic case (infinite correlation time τ). If the position of a parcel is white noise, it becomes impossible to represent the velocity, since the latter is undefined. Thus, solving the AC model...
analytically in the red noise case is a big step forward: the model becomes more general (the correlation time τ can be chosen arbitrarily and for instance be fit to the typical eddy turnover time) and physical (white noise is never observed in nature whereas red noise can represent turbulent eddies). We are especially interested in the new analytical expression for the bimodal PDF of moisture and the moisture flux.

More generally, we want to understand how the flux of a condensing scalar deviates from Fick’s law (Fick 1851) in the case of the OU process, which is a standard model for turbulent dispersion (Thomson 1987, 1990; Pasquero, Provenzale and Babiano 2001; Sawford 2001). According to Fick’s law, the flux of a scalar is proportional to its natural gradient, and in the opposite direction. Furthermore, if the scalar is strictly passive, the proportionality constant is the single-parcel diffusivity (the diffusivity of the parcel’s random motion itself). However, if a passive scalar (such as water vapor to a reasonable approximation) condenses and rains out in the atmosphere, the actual value of the moisture flux will be smaller than the value predicted by Fick’s law (see figure 2). The correlated AC model provides an analytical expression to quantify how the moisture flux deviates from its Fickian value, which is a central result of this paper.

2. A stochastic advection-condensation model

Following O’Gorman and Schneider (2006) and Sukhatme and Young (2011), we consider an idealized one-dimensional stochastic model for the advection and condensation of moisture (mass concentration of water vapor) \( q \). The parcels are in correlated random motion on an isentropic surface with meridional coordinate \( y \). The southern boundary of the domain is at \( y = 0 \), and the northern is at \( y = \ell \). The saturation specific humidity decreases monotonically from \( q_{\text{max}} \) at \( y = 0 \) to \( q_{\text{min}} \) at \( y = \ell \). A specific example used below is

\[
q_y(y) = q_{\text{max}} \exp(-\alpha y).
\]  

(1)

In this case the driest parcels are at \( y = \ell \) with specific humidity \( q_{\text{min}} = q_{\text{max}} \exp(-\alpha \ell) \).

2.1. Turbulent dispersion on an isentropic surface

The position \( Y(t) \) and velocity \( V(t) \) of a parcel on this surface are determined by the OU random flight model of turbulent dispersion:

\[
dY = V \, dt,
\]

(2)

\[
dV = -\frac{V}{\tau} + \sigma \sqrt{\frac{2}{\tau}} \, dW.
\]

(3)

Above we have introduced the correlation time \( \tau \), the root mean square velocity \( \sigma \) and the Wiener process \( W(t) \). Also, note that we use capital letters (eg \( Y \)) to denote random variables and small letters (eg \( y \)) to denote their values. The Lagrangian velocity autocorrelation function for positive times \((t_1, t_2) > 0\) is:

\[
E[V(t_1)V(t_2)] = \sigma^2 \exp\left(-\frac{|t_2-t_1|}{\tau}\right).
\]

(4)

The single-parcel diffusivity, which is the integral of \( E[V(0)V(t)] \) over positive \( t \), is therefore

\[
\kappa \overset{\text{def}}{=} \sigma^2 \tau.
\]

(5)

The equilibrium solution of this model is the well-known Maxwellian density (Doering 1990; Gardiner 2009):

\[
M(y, v) \overset{\text{def}}{=} \exp\left(-\frac{1}{2\pi \sigma^2 \ell} \right) \sqrt{2\pi \sigma^2 \ell}.
\]

(6)

2.2. The advection-condensation model

Each parcel carries a specific humidity \( Q(t) \), which decreases because of condensation \( C(Y, Q) \), and increases because of a moisture source \( S(Y) \). Thus in parallel with (3) the specific humidity of a parcel is determined by

\[
dQ = (S - C) \, dt.
\]

(7)

We model the condensation sink as follows: as soon as the moisture exceeds its saturation value, it is linearly decreased back to its saturation value

\[
C = \eta \left[ Q - q_y(Y) \right] H \left[ Q - q_y(Y) \right],
\]

(8)

where \( \eta \) is the condensation rate and \( q_y(x) \) is the prescribed saturation specific humidity; \( H(x) \) denotes the Heaviside step function, which is zero if \( x < 0 \) and one if \( x > 0 \). We work in the rapid-condensation limit, \( \eta \tau \ll 1 \). In this case (8) is replaced by a rule enforcing subsaturation:

\[
Q \rightarrow \min \left[ Q, q_y(Y) \right].
\]

(9)

The condensation rule (9), along with the assumption of a strictly decreasing saturation specific humidity profile \( q_y(y) \) restricts the ensemble of parcels to the shaded region of \((y, q)\) space depicted in figure 3.

Following Sukhatme and Young (2011), we model the source \( S \) by a “remoistening” at \( y = 0 \): when a southward moving parcel reaches \( y = 0 \) it is reflected back into the domain and its \( Q \) is reset by picking a new value from a probability density \( \Phi \).
2.3. The Fokker-Planck equation

In principle, the statistical properties of the moist-parcel model formulated above can be determined from a probability density function \( P(q,y,v) \), which is governed by the Fokker-Planck equation (Doering 1990; Gardiner 2009):

\[
\frac{\partial P}{\partial t} + \frac{\partial}{\partial q}[(S - C)P] + \frac{\partial (vP)}{\partial y} = \frac{1}{\tau} \frac{\partial}{\partial v}(vP + \sigma^2 \frac{\partial P}{\partial v}).
\] (10)

We can compare the OU Fokker-Planck equation (10) to its Brownian analog (e.g., equation (10) in Sukhatme and Young 2011). The diffusion term in displacement is replaced by the sum of an advection term and a red noise term, which allows us to define the velocity of the parcels, and make its profile Gaussian. The resetting moisture source \( S \) is specified by the boundary condition at \( y = 0 \), which is

\[
P(q,0,v > 0) = \Phi(q)M(y,v),
\] (11)

where \( M(y,v) \) is the equilibrium density in (6). Notice that the southern boundary condition is imposed only on parcels moving into the domain i.e., on that part of the density with \( v > 0 \). The “remoistening” distribution \( \Phi \) at \( y = 0 \) is prescribed in our model. Let’s consider the example of a moisture field \( q \) varying between \( q_{\text{min}} \) and \( q_{\text{max}} \). Then two simple distributions to test out the model are:

1. \( \Phi(q) = \delta^-(q - q_{\text{max}}) \), where all the parcels are remoistened to their maximal value \( q_{\text{max}} \) at \( y = 0 \). We define the Dirac delta functions \( \delta^+ \) and \( \delta^- \) in appendix A.
2. \( \Phi(q) = (q_{\text{max}} - q_{\text{min}})^{-1} \), where the parcels are uniformly remoistened.

At the northern boundary

\[
P(q,0,v < 0) = \delta^+(q - q_{\text{min}})M(y,v).
\] (12)

Again, the boundary condition is imposed only on that part of the density that corresponds to parcels moving into the domain. If we marginalize the Fokker-Planck equation (10) by integrating over \( q \) then we recover the simpler Fokker-Planck equation for the Ornstein-Uhlenbeck process, with equilibrium density \( M(y,v) \) in (6). Thus

\[
\int_{q_{\text{min}}}^{q_{\text{max}}} P(q,y,v) \, dq = M(y,v).
\] (13)

The goal of this paper is to solve the equilibrium version \((\partial P/\partial t = 0)\) of (10) through (12) and understand the statistical properties of the moisture distribution determined by the specified source at \( y = 0 \): \( \Phi(q) \), and by the condensation sink \( C \).

The steady version of the model is governed by one non-dimensional control parameter:

\[
L \overset{\text{def}}{=} \frac{\ell}{\tau \sigma}.
\] (14)

The length \( \sigma \tau \) is the typical distance a parcel moves before changing direction i.e., \( \sigma \tau \) is the distance moved in a correlation time \( \tau \). Thus the ratio \( L \) is a non-dimensional domain size. The “ballistic” limit considered by O’Gorman and Schneider (2006) corresponds to \( L \ll 1 \) and the Brownian limit of Sukhatme and Young (2011) is \( L \gg 1 \). Both cases are summarized in appendix E. Our goal is to solve the Fokker-Planck equation for all values of \( L \).

3. The two-stream model

Without the moisture variable \( q \), the solution of first-passage time problems for a correlated motion with reflective boundary conditions is very difficult. Even boundary-layer solutions, corresponding to \( L \gg 1 \), are intricate e.g., Hagan, Doering and Levermore (1989). The addition of the extra moisture variable \( q \) probably makes boundary-layer methods intractable.

Thus, instead of a direct assault on (10) we approach the problem by discretizing the Ornstein-Uhlenbeck process to make it simpler. The simplest case is the two-stream model in which we replace the velocity density \( v \) by two discrete velocities: \( \pm \sigma \). The PDF of three variables \( P(q,y,v) \) is replaced by two PDFs of two variables:

1. \( N(q,y) \), corresponding to the parcels moving Northwards at velocity \( +\sigma \).
2. \( S(q,y) \), corresponding to the parcels moving Southwards at velocity \( -\sigma \).

In order to mimic the red noise term in equation (10), the parcels are randomly switched between the two streams at a rate \( \beta/2 = \tau^{-1} \). The advection-exchange equations for \( (N,S) \) in the interior of the domain, where there are no sources and sinks \((N,S) = 0\), are:

\[
\frac{\partial}{\partial t} \begin{pmatrix} N \\ S \end{pmatrix} + \sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} N \\ S \end{pmatrix} = \frac{\beta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} N \\ S \end{pmatrix}.
\] (15)

3.1. Steady solution

We look for steady solutions \((\partial P/\partial t = 0)\) to equation (15). In the two-stream model, the control dimensionless parameter is:

\[
L \overset{\text{def}}{=} \frac{\ell}{\sigma \beta}.
\] (16)

There is an close analogy with (14) as \( \sigma \beta^{-1} \) is the typical distance a parcel moves before being exchanged and hence changing direction. We use this distance to make our variables dimensionless:

\[
y \overset{\text{def}}{=} \frac{\beta y}{\sigma}, \quad (\bar{N}, \bar{S}) \overset{\text{def}}{=} \frac{\sigma (N,S)}{\beta}.
\] (17)

For simplicity, we drop tildes in paragraphs 3.1, 3.2 and 3.3, where we exclusively work with dimensionless variables. Mathematically, working with dimensionless variables is equivalent to...
setting $(\alpha, \beta, \ell)$ to $(1, 1, L)$. If we marginalize the Fokker-Planck equation (15) over $q$, we expect the equilibrium solution to not have any gradients in $y$. The normalization condition for the PDFs $N$ and $S$ is then analogous to the normalization condition (13) for the Ornstein-Uhlenbeck process:

$$\int_{q_{\text{min}}}^{q_{\text{max}}} N(q, y) dq = \int_{q_{\text{min}}}^{q_{\text{max}}} S(q, y) dq = \frac{1}{2L}. \quad (18)$$

Note that the analog of the Maxwellian distribution $M(y, v)$ is the constant $(2L)^{-1}$. By construction of the two-stream model, the PDFs $N$ and $S$ have three parts, corresponding to three “groups” of parcels:

1. The parcels moving Northwards in a zone where $y > y_s(q)$ are supersaturated and condense as they move Northwards, constituting the saturated spike: $\delta^+ [q - q_s(y)]$.
2. The moisture of the parcels which have last hit the Northern boundary $y = L$ where $q = q_s(L) = q_{\text{min}}$ cannot be changed until the parcels hit the Southern boundary $y = 0$. These parcels constitute the dry spike: $\delta^+ (q - q_{\text{min}})$.
3. The rest of the PDF is called the smooth part.

The first step to computing the smooth part of the PDFs is to apply the normalized boundary condition at $y = 0$, analogous to (11):

$$N(q, 0) = \frac{\Phi(q)}{2L}. \quad (19)$$

Integrating the steady version of the two-stream Fokker-Planck equation (15) from $y' = 0$ to $y = y$ and using the boundary condition (19) gives the $y$-dependence of the smooth part and the dry spike of the vectorial PDF $(N^S)$:

$$\Phi(q) \left( \frac{1}{2L} \right) \left( \frac{1}{1} \right) + [C_{\text{smooth}}(q) + C_{\text{dry}} \delta^+ (q - q_{\text{min}})] \left( \frac{y}{y + 2} \right), \quad (20)$$

where $C_{\text{smooth}}(q)$ and $C_{\text{dry}}$ need to be determined. To proceed further, we note that by definition, the saturated spike is only contained in the PDF $N(q, y)$ of the Northwards moving particles. Thus, the PDF $S(q, y)$ of the Southwards moving particles only contains a dry spike and its smooth part: it is entirely given by (20). Writing its normalization condition (18) at $y = L$ where $q = q_s(L) = q_{\text{min}}$ yields:

$$\frac{1}{2L} = \int_{q_{\text{min}}}^{q_{\text{max}}} S(q, L) dq = C_{\text{dry}}(L + 2), \quad (21)$$

and allows us to find: $C_{\text{dry}} = (L + 2)^{-1}(2L)^{-1}$. To compute $C_{\text{smooth}}(q)$, we need to write the normalization condition (18) of $S(q, y)$ for all $y$:

$$\frac{1}{2L} = \frac{1 - \Lambda(q_s(y))}{2L} + (y + 2) \left[ \frac{1}{2L(L + 2)} + \int_{q_{\text{min}}}^{q_{\text{max}}} C_{\text{smooth}}(q) dq \right]. \quad (22)$$

We have used equation (20) to integrate $S(q, y)$ from $q_{\text{min}}$ to $q_s(y)$. Defined the cumulative distribution function (CDF) of the normalized distribution $\Phi(q)$:

$$\Lambda(q) \overset{\text{def}}{=} \int_q^{q_{\text{max}}} \Phi \quad (23)$$

such that $\Lambda(q_{\text{min}}) = 1$. (22) is an integral equation for $C_{\text{smooth}}(q)$: to solve it we change the independent variable from $y$ to $q$. Adapting the bounds of the integral by using figure 3, equation (22) becomes:

$$\frac{1}{2L} = \frac{1 - \Lambda(q)}{2L} + \left[ \frac{1}{2L(L + 2)} + \int_{q_{\text{min}}}^{q_{\text{max}}} C_{\text{smooth}}(q) dq \right]. \quad (24)$$

Differentiating equation (24) with respect to $q$ allows us to solve for $C_{\text{smooth}}(q)$:

$$C_{\text{smooth}}(q) = \frac{1}{2L} \frac{d}{dq} \left[ \frac{\Lambda(q)}{2L} + \int_{q_{\text{min}}}^{q_{\text{max}}} C_{\text{smooth}}(q) dq \right] \quad (25)$$

We are now left with the saturated peak of the PDF $N(q, y)$, that can be written $(2L)^{-1}W(y)\delta^+ [q - q_s(y)]$ and is derived in appendix B. Here, we simply compute the amount of supersaturated particles $W(y)$ by applying the normalization condition (18) of $N(q, y)$ for all $y$:

$$\frac{1}{2L} = \frac{1 - \Lambda(q_s)}{2L} + \frac{y}{y + 2} \frac{\Lambda(q_s)}{2L} + W(y) \frac{2\Lambda(q_s)}{y + 2} \quad (26)$$

$$W(y) = \frac{2\Lambda(q_s)}{y + 2} \quad (27)$$

Note that expression (27) is consistent with equation (49) derived in the appendix B defining $W(y)$. In summary, the rescaled vectorial PDF can be written:

$$2L \left( N(q, y) \frac{\Phi(q)}{2L} S(q, y) \right) = 2\Lambda(q_s) \left( \frac{\delta^+ (q - q_s)}{y + 2} \right) \quad (28)$$

Finally, following the condensation rule (9), the distribution vector is zero for $q > q_s(y)$. The main addition of correlating the velocity process is the saturated spike. In the Brownian limit (given by equation (79)), each parcel’s displacement is white noise, which prevents it from continuously moving northwards when it saturates. A minor addition is the fact that the dry spike is non zero at the Southern boundary of the domain $y = 0$. In the Brownian limit, every parcel close enough to the southern boundary hits it constantly, which remoistens it.

3.2. An example

Let’s consider the exponentially decreasing saturation specific humidity profile defined by (1). It could be an idealized representation of the Earth saturation specific humidity profile, if the temperature decreased approximately linearly with latitude from the Tropics to the Poles. We consider the case where the amount of condensed parcels $W(y)$ is maximal by assuming that the parcels are remoistened to their maximal humidity $q_{\text{max}}$ at $y = 0$:

$$\Phi(q) = \delta^+ (q - q_{\text{max}}) \quad (29)$$

Numerically, we run Monte-Carlo simulations by uniformly distributing $N = 10^6$ parcels on $y \in [0, L]$, and iterating the following steps until a statistically steady state is reached:

- Advecting the northwards and southwards parcels at velocities $\pm \sigma$.
- Reflecting the parcels at $y = 0$ and remoistening them to $q_{\text{max}}$.
- Reflecting the parcels at $q = L$ and drying them to $q_{\min}$.
- Enforcing the condensation rule: $q \leq q_{\text{max}}$.
- Exchanging the Northwards and Southwards parcels at a frequency $\frac{L}{2}$. 

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To compare the numerical results with the analytical expressions for the PDFs $N$ and $S$, we integrate the PDFs over a given $y$-strip $[y_1, y_2]$:

$$
\begin{pmatrix}
N_1(y) \\
S_1(y)
\end{pmatrix}
\text{def} = \int_{y_1}^{y_2} \begin{pmatrix}
N(q, y) \\
S(q, y)
\end{pmatrix} dy.
$$

(30)

The analytical result of integral (30) is given in appendix D and the comparison between this result and the numerical Monte-Carlo simulations is depicted in figure 4. For practical reasons, we only compare the smooth parts of the PDFs (28) and indicate the variations of $Q_N$ and $Q_S$, which have a specific universal form (59) for an isotropic and homogeneous velocity process (including the white noise and the ballistic limit). This universal form, determined by the definition of the advection-condensation model defined in section 2.2, is a good way to check the accuracy of the PDF. To understand the variations of $(N, S)$ with $y$, we also integrate the PDFs over a given $y$-strip $[y_1, y_2]$:

$$
\begin{pmatrix}
N_2(y) \\
S_2(y)
\end{pmatrix}
\text{def} = \int_{q_1}^{q_2} \begin{pmatrix}
N(q, y) \\
S(q, y)
\end{pmatrix} dq.
$$

(32)

The results of this integral are shown on figure 6 for different latitudes. Once again, we have only compared the smooth part of the PDFs and indicated the intensity of the saturated peak with an arrow. By definition, the dry peak can only be seen at $q = q_{\min} = 0.1$. The probability of finding a parcel of moisture $q$ is maximal at $y = y_s(q)$, where the Northwards parcels saturate to $q$. It is minimal near the Equator, where only the sub-saturated parcels that have been advected southwards from $y_s(q)$ to $y$ have a specific humidity equal to $q$. Because of the advective term of the FPE in the two-stream model (15), the probability increases linearly between these two limits. In the limit where $q_1 \to q_{\min}$ and $q_2 \to q_s(y)$, we obtain the northwards and southwards distributions of displacement given by equation (18).

### 3.3. Averages

Knowledge of the analytical expressions of the PDFs $N$ and $S$ allows us to evaluate the average of a function $f(q)$:

$$
\mathcal{T}(y) \text{def} = L \int_{q_{min}}^{q(y)} f(q)(N + S(q)) dq.
$$

(33)

Using (33):

$$
\mathcal{T} = \{\Lambda f\}(q_s) + \int_{q_{\min}}^{q_s(y)} \Phi f - (y + 1) \int_{q_{\min}}^{q_s(y)} \frac{df}{dy} \Lambda(q) dq dy + 2.
$$

(34)

We can compute the meridional gradient of the average by using Leibniz’s formula:

$$
\frac{d\mathcal{T}}{dy} = \frac{dq_s(\{\Lambda f\})(q_s)}{dy} - \int_{q_{\min}}^{q_s(y)} \frac{df}{dy} \frac{q_s(y) + 2}{y_s(q) + 2}.
$$

(35)

Applying (34) and (35) to the specific case $f(q) = q$ allows us to compute:

- The sub-saturation:

  $$
  q_{\text{sub}}(y) = q_s(y) - \mathcal{T}(y),
  $$

  (36)

and its average:

  $$
  \bar{q}_{\text{sub}} = \int_{q_{\min}}^{q_s(y)} \Phi q_{\text{sub}} + (y + 1) \int_{q_{\min}}^{q_s(y)} \frac{\Lambda(q) dq}{y_s(q) + 2}.
  $$

(37)
depicted on figure 7. Note that the average subsaturation is only zero at the southern boundary \( y = 0 \) in the Brownian limit: Any parcel close enough to the boundary will constantly hit it because of its white noise motion, and saturate as it is remoistened to its maximal value.

- The average meridional gradient of moisture:
  
  \[
  \frac{d\bar{q}}{dy} = \frac{dq_s}{dy} \frac{\Lambda(q_s)}{ys(q_s) + 2} - \int_{q_{\text{min}}}^{q_s(y)} \frac{\Lambda(q)dq}{ys(q) + 2}.
  \]  
  \( \text{(38)} \)

3.4. Diffusion

Coming back to dimensional variables, the flux of parcels is:

\[
\mathcal{F}_c \overset{\text{def}}{=} \sigma(N_{\text{dim}} - S_{\text{dim}}),
\]

where \((N_{\text{dim}}, S_{\text{dim}})\) are the dimensional PDFs in the two-stream model. From the advection-exchange equations, we can show that the flux of parcels verifies Fick’s law:

\[
\mathcal{F}_c = -\kappa \frac{\partial}{\partial y}(N_{\text{dim}} + S_{\text{dim}}),
\]

which confirms that the diffusivity of parcels is given by (5). The dimensional flux of moisture is defined as:

\[
\mathcal{F}_q \overset{\text{def}}{=} \ell \int_{q_{\text{min}}}^{q_s(y)} dq \sigma(N_{\text{dim}} - S_{\text{dim}})(q).
\]

(41)

Using the previous results, we can answer the question of how the flux deviates from Fick’s law by relating it to the moisture gradient; in dimensional notations:

\[
\mathcal{F}_q = -\kappa \frac{d\bar{q}}{dy} - W|\frac{dq_s}{dy}|.
\]

(42)

To first order, the flux of moisture is decreased by the amount \( W \) of parcels that have condensed weighted by the saturation moisture gradient.

3.5. Condensation rate

In our model, since the flux of moisture is zero at \( y = \ell \), the flux of moisture at a latitude \( y \) compensates the condensation occurring North of \( y \):

\[
\mathcal{F}_q = \ell \int_y^\ell dq \mathcal{C}(q, y')(N_{\text{dim}} + S_{\text{dim}})(q).
\]

(43)

\[
\mathcal{F}_q = \int_y^\ell \mathcal{C}(y')dy'.
\]

(44)

The average dimensional condensation rate is thus:

\[
\mathcal{C}(y) = -\frac{d\mathcal{F}_q}{dy} = \kappa \left( \frac{d^2\sigma q}{dz^2} + W^2 \frac{d^2q_s}{dy^2} - \frac{\beta}{\sigma} W \frac{dW}{dy} \right).
\]

(45)

4. Analytical approximation to the OU model

The \( n \)-stream model is the natural generalization of the two-stream model, where we consider \( n \) ensembles of parcels with \( n \) different velocities, that are exchanged at a rate proportional to \( \frac{q_{\text{rms}}^n}{2} \) (cf appendix F). As \( n \to +\infty \), the \( n \)-stream model converges to the OU model. Here, we use the analytical expressions derived for the average sub-saturation and the moisture flux in the case of the 2/3/4-stream model. We study how well they approximate the same quantities in the OU model, that we obtain from running Monte-Carlo simulations. The OU simulations are very similar to those described in 3.2 except that:

- Each parcel is advected by a velocity that follows an OU process with time-correlation \( \tau \).
- The parcels no longer need to be exchanged between different processes.

We have seen in 3.2 that the 2-stream model exactly reproduces the OU global distribution of moisture \( P_Q(q) \). In figure 7, the average sub-saturation is very well captured by the 2-stream model, except for the boundary layer near \( y = 0 \), which requires higher-order models, such as the 4-stream model. The values chosen for \((\kappa, \tau, \ell)\) give \( L = 1 \), and the OU model is quantitatively closer to the ballistic limit \( L \ll 1 \). As \( L \) increases, the OU average sub-saturation approaches the Brownian limit, and a maximum appears in the distribution, corresponding to a zone of minimal relative humidity. In figure 8, the moisture flux is also well-approximated by the 2/3/4-stream model, confirming the importance of taking into account the condensation in equation (42). Indeed, approximating the moisture flux as Fickian (eg in the Brownian limit) overestimates it by a factor ten near \( y = 0 \) for \( L = 1 \).

5. Conclusion and discussion

In this paper, we have studied the effect of condensation on a turbulent dispersion model, based on the Ornstein-Uhlenbeck process. For the sake of simplicity, we have introduced the \( n \)-stream models, which are discrete versions of the OU model. We
have obtained exact analytical solutions for the PDF, the moisture flux, the degree of subsaturation and the condensation rate of the 2/3/4-streams models. A central result of this paper is the local relation between the moisture flux and the moisture gradient, given by (42) in the two-stream model. This relation agrees with the result from (O’Gorman and Schneider 2006) in the Ballistic limit, and differs significantly from older parametrizations of the moisture flux (Stone and Yao 1990; Vallis 1982) that relied only on the average moisture or only on the saturation specific humidity. If more precision is needed, it is possible to use the 4-stream flux-gradient relation (114), although second derivatives of the mean moisture profile and of the saturation specific humidity profile can be hard to evaluate in practice. These results mean that if the dynamics of specific humidity are not resolved in a climate model, a proper parametrization of the moisture flux requires the knowledge of:

- The mean specific humidity gradient and the saturation specific humidity profile.
- The average amount of particles that have locally condensed, which additionally requires a basic knowledge of the eddy flow, such as the eddy turnover time and the eddy root mean square velocity.

It is important to note that this model relies on several key assumptions:

- The particles diffusivity is constant with latitude and the domain is Cartesian. Testing this model on reanalysis data would require us to generalize the two-stream model to a longitudinally symmetric sphere and to add a consistent latitudinal dependence of the diffusivity coefficient. It would be a direct way to quantify how much of the tropospheric moisture distribution is explained by advection and condensation in our current meteorological models.
- The only remoistening occurs at the Southern boundary of the domain: \( y = 0 \). Following (Sukhatme and Young 2011), this point could be made more physical by modeling the source \( S \) as a resetting \( y \rightarrow q_s(y) \) occurring at a rate \( \tau_{\text{moist}}(y) \) throughout the domain.
- The assumption of an exponentially decreasing saturation specific humidity profile \( q_s(y) \) comes from integrating the Clausius-Clapeyron equation assuming constant meridional temperature gradient and latent heat, as well as neglecting the partial pressure of water vapor compared to the total pressure. Except for \( P_2(q) \), the results of this paper do not change significantly when we choose a different function \( q_s(y) \) as long as it decreases monotonically. In reality, \( q_s \) is not monotonous and has strong local variations, which makes our model only valid on a large enough length scale and for a long enough timescale.
- The parcels evolve on surfaces of constant entropy. Taking the latent heating provided by condensation into account would result in saturated parcels following surfaces of constant saturated moist entropy instead of dry entropy. It would add a vertical structure to the problem, that we neglect here for the sake of simplicity.
- Finally, we have chosen the OU model for the velocity stochastic process, which imposes an exponentially decreasing auto-correlation function (4). This exponential behavior is only observed for motions generated by the eddy field on short timescales. In contrast, the auto-correlation function is closer to a power law for intermediate timescales (Sukhatme 2004), although many different dispersive regimes are observed in the atmosphere (Huber, McWilliams and Ghil 2001; Sherwood, Roca and Weckwerth 2010).

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### A. Dirac delta functions

The Dirac delta functions are distributions (generalized functions), defined as follows:

- \( \delta^+ \) is defined on the positive real number line \([0, +\infty]\) and is zero everywhere except at zero, with an integral of one over the entire positive real line.
- \( \delta^- \) is defined on the negative real number line \([-\infty, 0]\) and is zero everywhere except at zero, with an integral of one over the entire negative real line.

\[
\int_{-\infty}^{0} \delta^- \overset{\text{def}}{=} \int_{0}^{+\infty} \delta^+ \overset{\text{def}}{=} 1. \tag{46}
\]

### B. Saturated spike in the two-stream model

To understand how condensation occurs in the two-stream model, we relax the assumption of instant condensation (9); in dimensionless form equation (15) is generalized to:

\[
\frac{\partial}{\partial y} \left( \frac{N}{-S} \right) - \frac{\partial}{\partial \eta} \left[ C(q, y) \frac{N}{S} \right] = \frac{1}{2} \left( s - N \right) \tag{47}
\]

The dimensionless form of the condensation sink (8) is:

\[
C(q, y) = \eta^2 (q - q_s(y)) H[q(q - q_s(y))]. \tag{48}
\]

In the supersaturated region \( q \geq q_s(y) \), the parcels condense quickly enough for their velocities not to change sign, and we can assume \( S \ll N \). The second equation gives \( S \sim \eta \tau N \) making the approximation self-consistent, while integrating the first equation in the super-saturated region gives:

\[
\frac{dW}{dy} + \frac{W(y)}{2} = -\frac{d}{dy} N[q_s(y), y]. \tag{49}
\]

where we have defined the total density of supersaturated parcels:

\[
W(y) \overset{\text{def}}{=} \int_{q_s(y)}^{+\infty} N(q, y) dq. \tag{50}
\]

### C. Global distributions of displacement, velocity, and moisture

We work in the general case of the steady OU model (3), where the PDF in dimensionless variables is denoted by \( P(q, y, v) \). The global distributions of displacement, velocity, and moisture are respectively defined by:

\[
P_Y(y) \overset{\text{def}}{=} \int_{q_{\text{min}}}^{q_s(y)} dq \int_{-\infty}^{+\infty} dv P(q, y, v) \tag{51}
\]
\[
\begin{align*}
P_Y(v) & \overset{\text{def}}{=} \int_{q_{\text{min}}}^{q_s(y)} dq \int_0^L dy P(q, y, v) \tag{52} \\
P_Q(q) & \overset{\text{def}}{=} \int_{-\infty}^{+\infty} dq \int_0^L dy P(q, y, v) \tag{53}
\end{align*}
\]

The two first distributions can be obtained by integrating (13):

- In the steady case, the parcels are equally distributed along the meridional axis:
  \[P_Y(y) = L^{-1} \tag{54}\]
- By construction of the Gaussian colored noise, the distribution of velocities is Maxwellian:
  \[P_Y(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) \tag{55}\]

The global distribution of moisture as a function of \(\Phi(q)\) can be obtained from the fact that \(P_Y\) is constant:

\[\forall y, \quad P_Y(y) = \frac{1}{2} [P_Y(0) + P_Y(L)] \tag{56}\]

Integrating (56) from \(y' = 0\) to \(y' = y\):

\[\int_0^y dy' P_Y(y') = \frac{y}{2} [P_Y(0) + P_Y(L)] \tag{57}\]

Using the fact that \(P[q > q_s(y), y, v] = 0\), the definition of \(P_Y\), Fubini's theorem to exchange the integrals, switching the saturation variable from \(q\) to \(y\), differentiating with respect to \(q\), and using the definition of \(P_Q\) yields:

\[P_Q(q) = \frac{d}{dq} [y_s(q)] \int_{-\infty}^{+\infty} dq \int_{q_{\text{min}}}^{q_{\text{max}}} dq' P(q', 0, v) + P(q', L, v) \tag{58}\]

Combining (58) with the two normalized boundary conditions (11) and (12), we obtain:

\[P_Q(q) = \frac{1}{2} \left[ \delta^+(q - q_{\text{min}}) + \frac{d(y_s\Lambda)}{dq} \right] \tag{59}\]

- This demonstration is based on the fact that \(P_Y\) is constant, and is thus valid for any isotropic and homogeneous velocity stochastic process, including the n-stream process.
- The form of \(P_Q\) reflects the fact that half the parcels have just hit the dry boundary at \(y = L\) and constitute the dry spike, whereas the other half has last hit \(y = 0\) and has a moisture depending on the remoistening distribution and the saturation profile.

\section{PDF of the two-stream model in a strip}

\subsection{y-strip}

The PDFs \((N_{12}, S_{12})(y)\) integrated in a \(y\)-strip \([y_1, y_2]\) are defined by equation (32). Integrating the two-stream PDFs (28) is straightforward if we integrate each part separately:

- The smooth part:
  \[\int_{y_1}^{y_2} dq \frac{d}{dq} y_s(q) + \Lambda(q) + \left( y + 2 \right) + \left( \Phi(q) \Phi(q) \right) = \tag{66}\]
  \[\left( \frac{\Lambda(q)}{2L(y_s(q) + 2)} + \left( \frac{y + 2}{2} \right) + \left( \Phi(q) \Phi(q) \right) \right) \tag{67}\]

The slightly complicated form of \((s_{12}, t_{12})\) comes from the fact that the PDFs are 0 for \(q > q_s(y)\).

- The dry spike:
  \[\int_{y_1}^{y_2} \delta^+(q - q_{\text{min}}) dq = \tag{68}\]
  \[\frac{\Lambda(q)}{2L(y_s(q) + 2)} \delta[y_s(q_1) < y < y_s(q_2)] \tag{69}\]

where \(\delta[y_s(q_1) < y < y_s(q_2)]\) is equal to 1 if \(y_s(q_1) < y < y_s(q_2)\) and 0 otherwise.

- The saturated spike:
  \[\int_{y_1}^{y_2} \delta^+(q - q_{\text{min}}) dq = \tag{68}\]
  \[\frac{\Lambda(q)}{2L(y_s(q) + 2)} \delta[q_s(q_1) = y + 2] \tag{69}\]

where \(\delta[q_s(q_1) = y + 2]\) is equal to 1 if \(q_1 = q_{\text{min}}\) and 0 otherwise.

\section{D.2. q-strip}

The PDFs \((N_{12}, S_{12})(y)\) integrated in a \(q\)-strip \([q_1, q_2]\) are defined by equation (32). Integrating the two-stream PDFs (28) is straightforward if we integrate each part separately:

- The smooth part:
  \[\int_{q_1}^{q_2} dq \frac{d}{dq} y_s(q) + \Lambda(q) + \left( y + 2 \right) + \left( \Phi(q) \Phi(q) \right) = \tag{66}\]
  \[\left( \frac{\Lambda(q)}{2L(y_s(q) + 2)} + \left( \frac{y + 2}{2} \right) + \left( \Phi(q) \Phi(q) \right) \right) \tag{67}\]

The slightly complicated form of \((s_{12}, t_{12})\) comes from the fact that the PDFs are 0 for \(q > q_s(y)\).

- The dry spike:
  \[\int_{q_1}^{q_2} \delta^+(q - q_{\text{min}}) dq = \tag{68}\]
  \[\frac{\Lambda(q)}{2L(y_s(q) + 2)} \delta[y_s(q_1) < y < y_s(q_2)] \tag{69}\]

where \(\delta[y_s(q_1) < y < y_s(q_2)]\) is equal to 1 if \(y_s(q_1) < y < y_s(q_2)\) and 0 otherwise.

- The saturated spike:
  \[\int_{q_1}^{q_2} \delta^+(q - q_{\text{min}}) dq = \tag{68}\]
  \[\frac{\Lambda(q)}{2L(y_s(q) + 2)} \delta[q_s(q_1) = y + 2] \tag{69}\]

where \(\delta[q_s(q_1) = y + 2]\) is equal to 1 if \(q_1 = q_{\text{min}}\) and 0 otherwise.

\section{Summary}

\[\int_{y_1}^{y_2} dy \frac{dy_s\Lambda}{dq} \min\{y_s(q), y_s(q_2)\}^2 - \left( y_1 + 2 \right)^2 \tag{61}\]

\[t_{12}(q) = \min\{y_s(q), y_s(q_2)\} - y_1. \tag{62}\]
E. Limits of the OU model

E.1. The Ballistic limit

In the Ballistic limit \( L \ll 1 \), we rescale the latitude \( Y = L y \) to resolve the boundary layer, so that the FPE (10) becomes:

\[
L^{2} \frac{\partial P}{\partial Y} = \frac{\partial}{\partial y} (vP + \frac{\partial P}{\partial y}). \tag{70}
\]

To first approximation \( L \to 0 \) and:

\[
P(q, y, v) \approx \frac{r(q, v)}{\sqrt{2\pi L}} \exp\left(-\frac{v^{2}}{2}\right), \tag{71}
\]

where \( r \) is a function satisfying \( \frac{\partial r}{\partial m} = 0 \) almost everywhere. Physically, the velocity of a parcel remains unchanged from one boundary to another, which allows us write \( r \) as a sum of:

- A northwards part, using the fact that the parcels saturate continuously as they move Northwards:
  \[
  \{ \Phi(q)H[q_{s}(y) - q] + \delta[q - q_{s}(y)]\Lambda(q) \} H(v). \tag{72}
  \]
  We have used the definition (23).
- A Southwards part, which reduces to a dry spike:
  \[
  \delta^{+}(q - q_{\text{min}})H(-v). \tag{73}
  \]

In this limit, the average of a function \( f(q) \) is defined by:

\[
\overline{f}(y) \overset{\text{def}}{=} L \int_{-\infty}^{+\infty} dv \int_{q_{\text{min}}}^{q_{s}(y)} f(q)P(q, y, v)dq. \tag{74}
\]

From equations (74) and (71), we can approximate:

- The sub-saturation as:
  \[
  q_{\text{sub}}(y) \approx q_{s}(y) - q_{\text{min}} - \frac{1}{2} \int_{q_{\text{min}}}^{q_{s}(y)} \Lambda. \tag{75}
  \]
- The moisture flux as:
  \[
  F_{q} \approx \sqrt{2\pi} \sigma \overline{\Phi}(y). \tag{76}
  \]

E.2. The Brownian limit

In the Brownian limit \( L \gg 1 \), the stochastic differential equations for the displacement (3) reduces to:

\[
dY(t) = \sqrt{2\kappa}dW(t), \tag{77}
\]

making the FPE a Laplace equation for the PDF:

\[
\frac{\partial^{2}P}{\partial y^{2}} = 0. \tag{78}
\]

The detailed solution of the Brownian problem can be found in (Sukhatme and Young 2011); the main results of interest in our case are:

- The PDF:
  \[
  P(q, y) = \frac{1}{L} \{ \Phi(q) + y \frac{d}{dy} \frac{\Lambda(q)}{y_{s}(q)} + \delta^{+}(q - q_{\text{min}}) \}. \tag{79}
  \]

- The average sub-saturation:
  \[
  q_{\text{sub}}(y) = \int_{q_{\text{min}}}^{q_{s}(y)} \Phi q_{\text{sub}} + y \int_{q_{\text{min}}}^{q_{s}(y)} \Lambda dq \frac{y_{\text{sub}}}{y_{s}(q)}. \tag{80}
  \]

- The Fickian moisture flux:
  \[
  F_{q} = -\kappa \frac{\partial}{\partial y}. \tag{81}
  \]

F. The n-stream model

F.1. Definition

The n-stream model is a natural discretization of the OU model (3) and thus a generalization of the two-stream model (15). It is easy to think about it as \( n \) bits, which can take the value \( \pm 1 \). If we consider one combination of bits, the sum of the bits gives the velocity of the corresponding parcel’s ensemble, which determines the advection matrix of the process: \( A \). The exchange of parcels between ensembles happens when one bit’s value is modified, and the probability of switching from one velocity to another gives the exchange matrix of the process: \( E \). Finally, the normalization condition is obtained by considering the probability of a combination of bits. Mathematically, defining the vectorial PDF \( \mathbf{P}(q, y) \), the equations defining the n-stream model are:

\[
\frac{\partial}{\partial t} \mathbf{P} + \frac{\sigma}{\sqrt{n-1}} A \frac{\partial}{\partial y} \mathbf{P} = \beta \mathbf{E} \mathbf{P} \tag{82}
\]

\[
A_{ij} = (n + 1 - 2i) \delta_{ij} \tag{83}
\]

\[
E_{ij} = -n \delta_{ij} + n \delta_{i+1,j} + (n - j) \delta_{i-1,j}, \tag{84}
\]

where \( \delta_{ij} \) is the Kronecker symbol. The normalization condition for \( \mathbf{P}(q, y) \) yields:

\[
\int_{q_{\text{min}}}^{q_{s}(y)} P_{i}(q, y)dq = \frac{(n - 1)!}{(i - 1)! (n - i)!} \frac{1}{2^{n-1}L}. \tag{85}
\]

where \( i \) is the index of the vector \( \mathbf{P}(q, y) \).

F.2. The three-stream model

The steady three-stream equations can be written:

\[
\frac{\sigma}{\sqrt{2}} \frac{\partial}{\partial y} \begin{pmatrix} 2N \\ 0 \\ -2S \end{pmatrix} = \frac{\beta}{2} \begin{pmatrix} -2 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} N \\ M \\ S \end{pmatrix}, \tag{86}
\]

where \( (N, M, S) \) respectively represent the Northwards, motionless and Southwards parcel’s PDFs. We now work with dimensionless variables. The normalization condition for the vectorial PDF is:

\[
\int_{q_{\text{min}}}^{q_{s}(y)} \begin{pmatrix} N \\ M \\ S \end{pmatrix} dq = \frac{1}{4L} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{87}
\]

Following the same steps as for the two-stream model, we find that the vectorial PDF is the sum of:

- A smooth part:
  \[
  \frac{\Phi(q)}{4L} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + C_{\text{smooth}}(q) \begin{pmatrix} 2(q + \sqrt{2}) \\ (y + \sqrt{2}) \end{pmatrix}. \tag{88}
  \]

- A dry spike:
  \[
  C_{\text{dry}} \begin{pmatrix} y \\ 2(y + \sqrt{2}) \end{pmatrix}. \tag{89}
  \]

- A saturated spike:
  \[
  \frac{W(y)}{4L} \delta[q - q_{s}(y)] \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{90}
  \]
where from the boundary and normalization conditions:
\[
C_{\text{dry}} = \frac{1}{4L(L + 2\sqrt{2})}
\]
\[
C_{\text{smooth}}(q) = \frac{1}{4L} \frac{d}{dq} \frac{\Lambda(q)}{y_s(q) + 2\sqrt{2}}
\]
\[
W(y) = \frac{\sqrt{2} \Lambda(q_s(y))}{y + 2\sqrt{2}}.
\]  
(91)  
(92)  
(93)

Defining the average of a function \( f(q) \) by:
\[
\overline{f}(y) \equiv L \int_{q_{\min}}^{q(y)} f(q)(N + M + S)(q) dq.
\]  
(94)

the average sub-saturation is:
\[
\overline{\theta}_{\text{sub}} = \int_{q_{\min}}^{q(y)} \Phi_{\text{sub}} + (y + \sqrt{2}) \int_{q_{\min}}^{q(y)} \frac{\Lambda(q) dq}{y_s(q) + 2\sqrt{2}}.
\]  
(95)

Defining the dimensional moisture flux as:
\[
\mathcal{F}_q \equiv \ell \int_{q_{\min}}^{q(y)} \sqrt{2} \sigma q(N - S)(q) dq,
\]  
(96)

we can once again relate it to the moisture gradient in dimensional form:
\[
\mathcal{F}_q = -\kappa \left( \frac{\partial \mathcal{G}_q}{\partial y} - W \frac{dq}{dy} \right).
\]  
(97)

There is very little difference between the 2/3-stream models, except for the presence of motionless parcels, which explains why the moisture flux is smaller in the three-stream case.

F.3. The four-stream model

The steady four-stream equations can be written:
\[
\frac{\sigma}{\sqrt{3}} \frac{\partial}{\partial y} \left( \begin{array}{c}
3N_3 \\
N_1 \\
-N_1 \\
-S_3 \end{array} \right) = \frac{\beta}{2} \left( \begin{array}{cccc}
-3 & 1 & 0 & 0 \\
3 & -3 & 2 & 0 \\
0 & 2 & -3 & 3 \\
0 & 0 & 1 & -3 \end{array} \right) \left( \begin{array}{c}
N_3 \\
N_1 \\
S_1 \\
S_3 \end{array} \right),
\]  
(98)

where \((N_3, N_1, S_1, S_3)\) respectively represent the fast northwards, slow northwards, slow southwards and fast southwards parcel's PDFs. We now work with dimensionless variables. The normalization condition for the vectorial PDF is:
\[
\int_{q_{\min}}^{q(y)} \left( \begin{array}{c}
N_3 \\
N_1 \\
S_1 \\
S_3 \end{array} \right) dq = \frac{1}{8\ell} \left( \begin{array}{c}
1 \\
3 \\
3 \\
1 \end{array} \right).
\]  
(99)

Following the same steps as for the 2/3-stream models, we find that the vectorial PDF is the sum of:

- A smooth part:
\[
\phi(q) = \frac{1}{8\ell} \left( \begin{array}{c}
1 \\
3 \\
3 \\
1 \end{array} \right) + 147\sqrt{3} C_{\text{smooth}}(q) \zeta_{\pm}(y).
\]  
(100)

- A dry spike:
\[
147\sqrt{3} C_{\text{dry}, \pm}(q) \zeta_{\pm}(y).
\]  
(101)

- A saturated spike:
\[
\frac{\delta^+}{8\ell} \left( \begin{array}{c}
W_3(y) \\
W_1(y) \\
0 \\
0 \end{array} \right).
\]  
(102)

where a sum over \( \pm \) is implied and \( \zeta_{\pm}(y) \) is given by:
\[
\left( \begin{array}{c}
1 \pm 6\sqrt{2} \pm \sqrt{6}(y + 2\sqrt{3}) - \exp(\pm \sqrt{6}y) \\
3(1 \pm 6\sqrt{2}) \pm \sqrt{6}(2y + 8\sqrt{3}) - 3(1 \pm 2\sqrt{2}) \exp(\pm \sqrt{6}y) \\
3(1 \pm 6\sqrt{2}) \pm \sqrt{6}(3y + 10\sqrt{3}) - 3(1 \pm 4\sqrt{2}) \exp(\pm \sqrt{6}y) \\
(1 \pm 6\sqrt{2}) \pm \sqrt{6}(y + 4\sqrt{3}) + (3 \pm 2\sqrt{2}) \exp(\pm \sqrt{6}y) \end{array} \right).
\]  
(103)

From the boundary and normalization conditions, we can solve for the six unknowns \( C_{\text{smooth}, \pm}(q), C_{\text{dry}, \pm}, W_1(y) \) and \( W_3(y) \):
\[
C_{\text{dry}, \pm} = -\frac{1}{8\ell} \sqrt{3} \Gamma(L),
\]  
(104)

\[
C_{\text{smooth}, \pm}(q) = -\frac{1}{8\ell} \sqrt{3} \frac{d}{dq}(\Gamma_{\pm}[y_s]),
\]  
(105)

\[
W_3(y) \frac{\mathcal{W}(y)}{\mathcal{W}(y)} = -\sqrt{2} + 5\sqrt{2} \cosh(\sqrt{6}y) + 8 \sinh(\sqrt{6}y),
\]  
(106)

\[
W_1(y) \frac{\mathcal{W}(y)}{\mathcal{W}(y)} = 3(\sqrt{2} + 3\sqrt{2} \cosh(\sqrt{6}y) + 4 \sinh(\sqrt{6}y)),
\]  
(107)

where \( \Gamma_{\pm}(y) \) and \( \mathcal{W}(y) \) are respectively defined and given by:
\[
-\sqrt{2} + (3\sqrt{2} \pm 4) \exp(\sqrt{6}y)
\]  
(108)

\[
-\sqrt{2} + \sqrt{2}(9 + 2\sqrt{3}) \cosh(\sqrt{6}y) + (14 + 3\sqrt{3}) \sinh(\sqrt{6}y)
\]  
(109)

Defining the average of a function \( f(y) \) by:
\[
\overline{f}(y) \equiv L \int_{q_{\min}}^{q(y)} f(q)(N_3 + N_1 + S_1 + S_3)(q) dq.
\]  
(110)

the average sub-saturation is:
\[
\overline{\theta}_{\text{sub}} = \int_{q_{\min}}^{q(y)} \Phi_{\text{sub}} - Q(y).
\]  
(111)

We have defined:
\[
Q(y) \equiv \frac{1}{4} \pm \sqrt{6}(\sqrt{3} + y) - (1 \pm \sqrt{2}) \exp(\pm \sqrt{6}y) \int_{q_{\min}}^{q(y)} \Gamma_{\pm}[y_s],
\]  
(112)

where a sum over \( \pm \) is implied. Defining the moisture flux as:
\[
\mathcal{F}_q \equiv \ell \int_{q_{\min}}^{q(y)} \sqrt{3} \sigma q(N_3 + N_1 - S_1 - S_3)(q) dq.
\]  
(113)

we need to use the first and second meridional derivative of the average moisture to write the gradient/flux relation:
\[
\mathcal{F}_q = -\kappa \left( \frac{\partial \mathcal{G}_q}{\partial y} - W_{\text{tot}} \frac{dq}{dy} \right)(1 - \frac{2}{D})
\]  
(114)

\[
+ \kappa \ell \frac{d^2 q}{dy^2} - W_{\text{tot}} \frac{d^2 q}{dy^2} \left(1 - \frac{2 + (\sqrt{2} - 1) \exp(-\sqrt{6}y)}{D} \right),
\]

where \( W_{\text{tot}} = W_1 + W_3 \) is the total amount of parcels which have condensed, \( L = 6 \mp \beta \sigma \) is the boundary layer decay-length of the 4-stream model and the denominator in dimensionless form is:
\[
D(y) \equiv 2 + \cosh(\sqrt{6}y) + \sqrt{2} \sinh(\sqrt{6}y)
\]  
(115)

Qualitatively, the 4-stream model adds a boundary layer structure to the solution, which explains the better agreement of the analytical 4-stream average subsaturation with the OU average subsaturation for small \( y \) (cf figure 7). Furthermore, the boundary layer provides a positive contribution depending on \( \frac{d^2 q}{dy^2} \) to the moisture flux, once again decreased by the amount of condensed parcels.
References


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